

A note on the Grcar matrix

G rard MEURANT

<https://gerard-meurant.fr>

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This matrix was proposed by **Joseph F. Garcia** in 1989 to test some iterative methods

For $n = 10$, the matrix is

$$G_{10} = \begin{pmatrix} 1 & 1 & 1 & 1 & & & & & & \\ -1 & 1 & 1 & 1 & 1 & & & & & \\ & -1 & 1 & 1 & 1 & 1 & & & & \\ & & -1 & 1 & 1 & 1 & 1 & & & \\ & & & -1 & 1 & 1 & 1 & 1 & & \\ & & & & -1 & 1 & 1 & 1 & 1 & \\ & & & & & -1 & 1 & 1 & 1 & 1 \\ & & & & & & -1 & 1 & 1 & 1 \\ & & & & & & & -1 & 1 & 1 \\ & & & & & & & & -1 & 1 \end{pmatrix}$$

In fact, the matrix defined by Grcar was the transpose of that matrix, but this does not matter for our purposes

The matrix G_n of order n is Toeplitz upper Hessenberg and banded with -1 on the subdiagonal, 1 on the main diagonal, and on three upper diagonals

It is a nonderogatory matrix

We are interested in the determinant, the inverse, factorizations, and the eigenvalues

Determinant

The determinants $d_n = \det(G_n)$ are

$$d_1 = 1,$$

$$d_2 = d_1 + 1 = 2,$$

$$d_3 = d_2 + d_1 + 1 = 4,$$

$$d_4 = d_3 + d_2 + d_1 + 1 = 8,$$

$$d_n = d_{n-1} + d_{n-2} + d_{n-3} + d_{n-4}, \quad n \geq 5.$$

The determinants are always positive, and increasing with n . The first values are

$$1, 2, 4, 8, 15, 29, 56, 108, 208, 401,$$

for $n = 1, 2, \dots, 10$

The value of the determinant is approximately multiplied by 2 when n is increased by 1. We have $\det(G_{100}) \approx 1.7944 \cdot 10^{28}$ and $\det(G_{200}) \approx 5.6852 \cdot 10^{56}$

Inverse

Let P be the permutation matrix corresponding to

$$(1, 2, 3, \dots, n) \rightarrow (2, 3, \dots, n, 1)$$

Let H be an unreduced upper Hessenberg matrix of order n such that

$$PH = \begin{pmatrix} \hat{H} & w \\ h^T & h_{1,n} \end{pmatrix}$$

where \hat{H} is square of order $n - 1$ and upper triangular. Let $\ell^T = h^T \hat{H}^{-1}$ and $\alpha = h_{1,n} - \ell^T w$. The inverse of PH is

$$(PH)^{-1} = \begin{pmatrix} \hat{H}^{-1}(I + \frac{1}{\alpha} w \ell^T) & -\frac{1}{\alpha} \hat{H}^{-1} w \\ -\frac{1}{\alpha} \ell^T & \frac{1}{\alpha} \end{pmatrix}$$

and

$$H^{-1} = \frac{1}{\alpha} \begin{pmatrix} -\hat{H}^{-1} w & \hat{H}^{-1}(\alpha I + w \ell^T) \\ 1 & -\ell^T \end{pmatrix}$$

We apply this result to the Grcar matrix

The diagonal entries of the matrix \widehat{G}_n of order $n - 1$ are equal to -1 and there are three upper diagonals with entries equal to 1.

For instance, for $n = 10$,

$$\widehat{G}_{10} = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & & & & & \\ & -1 & 1 & 1 & 1 & 1 & & & & \\ & & -1 & 1 & 1 & 1 & 1 & & & \\ & & & -1 & 1 & 1 & 1 & 1 & & \\ & & & & -1 & 1 & 1 & 1 & 1 & \\ & & & & & -1 & 1 & 1 & 1 & \\ & & & & & & -1 & 1 & 1 & \\ & & & & & & & -1 & 1 & \\ & & & & & & & & -1 & \\ & & & & & & & & & -1 \end{pmatrix}$$

The inverse of \widehat{G}_n is a Toeplitz upper triangular matrix

$$(\widehat{G}_n)^{-1} = \begin{pmatrix} -1 & -d_1 & -d_2 & -d_3 & \cdots & -d_{n-3} & -d_{n-2} \\ & -1 & -d_1 & -d_2 & & -d_{n-4} & -d_{n-3} \\ & & -1 & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & -1 & -d_1 & -d_2 \\ & & & & & -1 & -d_1 \\ & & & & & & -1 \end{pmatrix}$$

where the d_i 's are the determinants given above

Using the previous notation

$$h^T = (1 \quad 1 \quad 1 \quad 1 \quad 0 \quad \cdots \quad 0)$$

and, for $n > 4$, w is a zero vector except for the last four components equal to 1. Therefore

$$\begin{aligned} \ell^T &= h^T (\widehat{G}_n)^{-1}, \\ &= (-1, \quad -1 - d_1, \quad -1 - d_1 - d_2, \quad -1 - d_1 - d_2 - d_3, \quad \cdots \\ &\quad -d_{n-5} - d_{n-4} - d_{n-3} - d_{n-2}), \\ &= (-d_1, \quad -d_2, \quad -d_3, \quad \cdots \quad -d_{n-1}) \end{aligned}$$

Since, for $n > 4$, $h_{1,n} = 0$, $\alpha = -\ell^T w$, that is, the sum of the last four components of ℓ^T , which yields $\alpha = -d_n$

For $(\widehat{G}_{n-1})^{-1}w$, we just have to compute the sum of the four last entries of each row of the matrix. It yields

$$(\widehat{G}_{n-1})^{-1}w = (-d_{n-1} \quad -d_{n-2} \quad \cdots \quad -d_2 \quad -d_1)^T$$

The entries of the rank-one matrix $(\widehat{G}_{n-1})^{-1}w \ell^T$ are products of determinants

and U_n upper triangular with entries $u_{i,j}$

$$\begin{aligned}u_{1,1:4} &= (1 \quad 1 \quad 1 \quad 1), \\u_{2,2:5} &= \left(\frac{d_2}{d_1} \quad \frac{d_3-d_2}{d_1} \quad 1 + \frac{1}{d_1} \quad 1 \right)\end{aligned}$$

and, for $k \geq 3$,

$$\begin{aligned}u_{k,k} &= \frac{d_k}{d_{k-1}}, \\u_{k,k+1} &= \frac{d_{k+1} - d_k}{d_{k-1}}, \quad k+1 \leq n, \\u_{k,k+2} &= 1 + \frac{d_{k-2}}{d_{k-1}}, \quad k+2 \leq n, \\u_{k,k+3} &= 1, \quad k+3 \leq n, \\u_{k,j} &= 0, \quad j > k+3 \text{ when } k+3 < n.\end{aligned}$$

Eigenvalues

As far as we know, there is no explicit formula for the eigenvalues of Grcar matrices

In 1985, W.F. Trench published a characterization of the eigenvalues of banded Toeplitz matrices

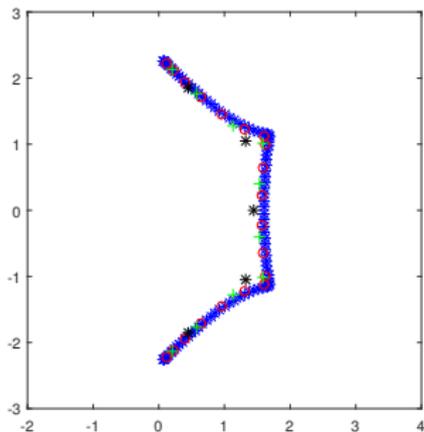
Let $q(z, \lambda) = z^4 + z^3 + z^2 + (1 - \lambda)z - 1$ be the polynomial defined by the symbol of $G_n - \lambda I$ after multiplication by z

For a given λ , the polynomial $q(z, \lambda)$ has distinct roots except for at most four values of λ

The generic case is that $q(z, \lambda) = 0$ has four distinct solutions z_i , $i = 1, \dots, 4$. Let

$$Z_n = \begin{pmatrix} 1 & 1 & 1 & 1 \\ z_1^{n+1} & z_2^{n+1} & z_3^{n+1} & z_4^{n+1} \\ z_1^{n+2} & z_2^{n+2} & z_3^{n+2} & z_4^{n+2} \\ z_1^{n+3} & z_2^{n+3} & z_3^{n+3} & z_4^{n+3} \end{pmatrix}$$

whose entries are functions of λ . Trench proved that λ is in the spectrum of G_n if and only if $\det(Z_n) = 0$



Eigenvalues of Grcar matrices G_n , $n = 5$ (black *), $n = 10$ (green +), $n = 20$, (red o), $n = 100$, (blue *)

When n is large, the eigenvalues of G_n are located on well-defined curves and there is some kind of convergence

Asymptotic spectrum

The limit of spectra of Toeplitz matrices was investigated by P. Schmidt and F. Spitzer in 1960

They introduced the concept of asymptotic spectrum. Let the spectrum be

$$\Sigma_n = \{\lambda \mid \det(G_n - \lambda I) = 0\}.$$

The asymptotic spectrum is defined as

$$\Sigma_a = \{\lambda \mid \lambda = \lim_{m \rightarrow \infty} \lambda_m, \lambda_m \in \Sigma_{\ell_m}, \lim_{m \rightarrow \infty} \ell_m = \infty\}.$$

It means that there is a subsequence of spectra converging in some sense to the asymptotic spectrum

In general, Σ_n is not contained in Σ_a for all values of n .

Schmidt and Spitzer proved that Σ_a is not empty, has no isolated point, and consists of a finite number of analytic arcs

However, they did not prove that it is connected. This was proved by J.L. Ullman in 1967

I.I. Hirschman Jr. proved in 1967 that Σ_a can be represented as a finite union of closed analytic arcs, where either distinct arcs are disjoint, or, if not, their intersection consists of one or both common end points. He studied the limiting eigenvalue distribution. He showed that there exists a probability measure μ on Σ_a such that

$$\frac{1}{n} \sum_{\lambda \in \Sigma_n} \delta_\lambda \rightarrow \mu,$$

where each eigenvalue in the sum is counted according to its multiplicity

From Schmidt and Spitzer we know that Σ_a is the set of λ 's in the complex plane such that the two roots of

$$\lambda = -\frac{1}{z} + 1 + z + z^2 + z^3$$

with the two smallest moduli have the same modulus

Since the entries of G_n are real, Σ_a is symmetric with respect to the x -axis

This equation can be converted into a polynomial equation of degree 4 in z or $1/z$ similar to $q(z, \lambda)$

In 1993, R.M. Beam and R.F. Warming proposed an algorithm for computing points on the asymptotic spectrum of a general banded Toeplitz matrix

However, for G_n we can obtain a parametric description of the asymptotic spectrum

We write $z = \hat{z} e^{-i\phi}$ with $0 < \phi < \pi$

Since $\hat{z} e^{i\phi}$ must give the same value of λ , by subtracting, we obtain

$$\sin(\phi) \frac{1}{\hat{z}} + \sin(\phi) \hat{z} + \sin(2\phi) \hat{z}^2 + \sin(3\phi) \hat{z}^3 = 0.$$

It is easier to work with $y = 1/\hat{z}$ for which we obtain

$$y^4 + y^2 + \frac{\sin(2\phi)}{\sin(\phi)} y + \frac{\sin(3\phi)}{\sin(\phi)} = 0.$$

But, $\sin(2\phi) = 2 \sin(\phi) \cos(\phi)$ and $\sin(3\phi) = 3 \sin(\phi) - 4 \sin^3(\phi)$

Defining $\zeta = \cos(\phi)$, the equation becomes

$$y^4 + y^2 + 2\zeta y - 1 + 4\zeta^2 = 0$$

This is a depressed quartic equation

The nature of the roots (real or complex) depends on the sign of the discriminant Δ

$$\Delta(\zeta) = 48\zeta^2 - 432\zeta^4 - 16 - 128(4\zeta^2 - 1)^2 + 576\zeta^2(4\zeta^2 - 1) + 256(4\zeta^2 - 1)^3$$

Δ has only two real zeros, symmetric with respect to 0, one negative and one positive which is approximately

$$\zeta_+ = 0.5623291174585$$

$\Delta(\zeta)$ is negative when $\zeta \in (\zeta_-, \zeta_+)$, and positive outside

The equation for y has two complex conjugate pairs of roots when

$\Delta(\zeta) > 0$ and two distinct real roots and a complex pair when

$\Delta(\zeta) < 0$

We first consider the case $\Delta(\zeta) > 0$, that is, $\zeta > \zeta_+$. We have two pairs of complex conjugate roots

The four solutions can be written as

$$r + \imath s_1, \quad r - \imath s_1, \quad -r + \imath s_2, \quad -r - \imath s_2$$

with $r = \sqrt{\hat{r}}$

$$\hat{r} = 2\sqrt{\frac{\zeta^2}{3} - \frac{33}{432}} \cos\left(\frac{1}{3} \arccos\left(\frac{\frac{111}{5184} - \frac{5\zeta^2}{96}}{\left(\frac{\zeta^2}{3} - \frac{33}{432}\right)^{\frac{3}{2}}}\right)\right) - \frac{1}{6}$$

and

$$s_1 = \sqrt{\frac{1}{2} \left(1 + 2\hat{r} + \frac{\zeta}{r}\right)}$$

We can assume that the solutions have this form because the coefficient of y^3 is zero

Taking the product of the four roots, and by identification with the coefficients of the polynomial, we have

$$(A) \quad (r^2 + s_1^2) + (r^2 + s_2^2) = 1 + 4r^2,$$

$$(B) \quad (r^2 + s_1^2) - (r^2 + s_2^2) = \frac{\zeta}{r},$$

$$(C) \quad (r^2 + s_1^2)(r^2 + s_2^2) = 4\zeta^2 - 1$$

We need to have $1 \geq \zeta \geq 1/2$ which is satisfied with our hypothesis on ζ . Using (A)+(B), (A)-(B), and (C), we obtain an equation for r

$$16r^6 + 8r^4 + (5 - 16\zeta^2)r^2 - \zeta^2 = 0$$

Using $\hat{r} = r^2$, we have a cubic polynomial equation that we solve for \hat{r} as a function of ζ

For the case $\Delta(\gamma) \leq 0$, that is, $\zeta \in (0, \zeta_+]$, we can show that we obtain the same equation for r

We already know that we have two distinct real roots and a pair of complex conjugate roots if $\zeta < \zeta_+$

The two solutions with the same modulus are those of the complex pair $r \pm is$

Let

$$\alpha = \frac{33}{432} - \frac{\zeta^2}{3}, \quad \beta = \frac{111}{5184} - \frac{5\zeta^2}{96}, \quad \gamma = \alpha^3 + \beta^2$$

$$\delta = (|\beta| + \sqrt{\gamma})^{1/3}$$

$$\hat{r} = \begin{cases} \delta - \frac{\alpha}{\delta} - \frac{1}{6} & \text{if } \beta \geq 0, \\ \frac{\alpha}{\delta} - \delta - \frac{1}{6} & \text{if } \beta < 0 \end{cases}$$

and $s = \sqrt{\frac{1}{2} \left(1 + 2\hat{r} + \frac{\zeta}{r} \right)}$ with $r = \sqrt{\hat{r}}$

If $\zeta = \zeta_+$, the two real solutions are equal. Their modulus is smaller than the moduli of the roots in the complex pair

To obtain the upper part (above the x -axis) of the asymptotic spectrum, we use the solution $r - \imath s$ with $s = s_1$ described previously

We have to multiply with $e^{\imath\phi} = \zeta + \imath\sqrt{1 - \zeta^2}$ to get

$$\frac{1}{z} = \tilde{y} = (r\zeta + s\sqrt{1 - \zeta^2}) + \imath(-s\zeta + r\sqrt{1 - \zeta^2})$$

Note that we have $|\tilde{y}|^2 = r^2 + s^2$

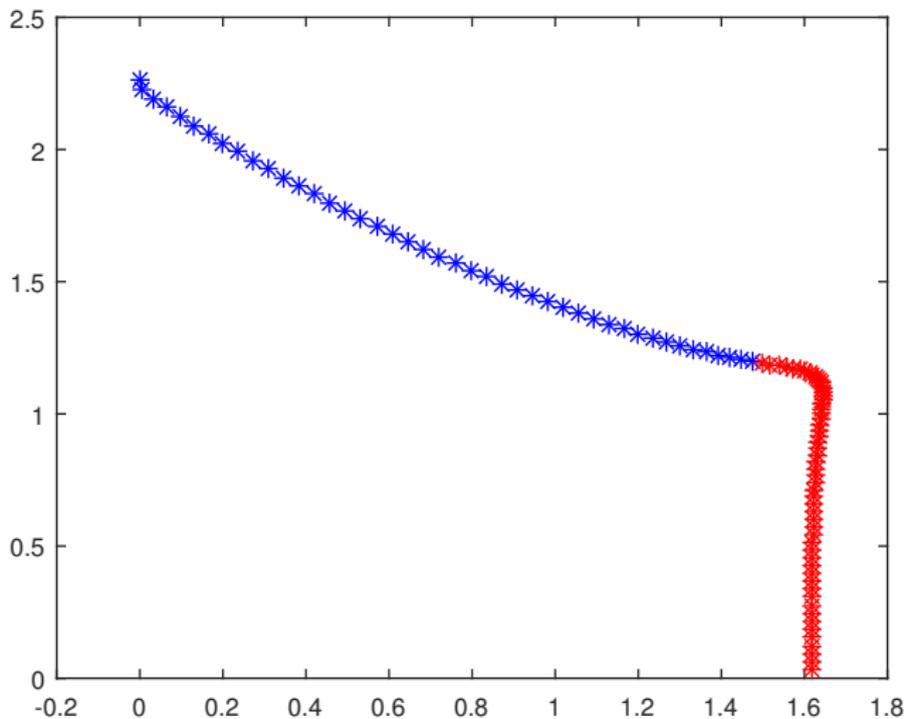
Let $\zeta \in (0, 1)$ and

$$\sin(\varphi) = \frac{-s\zeta + r\sqrt{1-\zeta^2}}{\sqrt{r^2 + s^2}} < 0, \quad \cos(\varphi) = \frac{r\zeta + s\sqrt{1-\zeta^2}}{\sqrt{r^2 + s^2}} > 0.$$

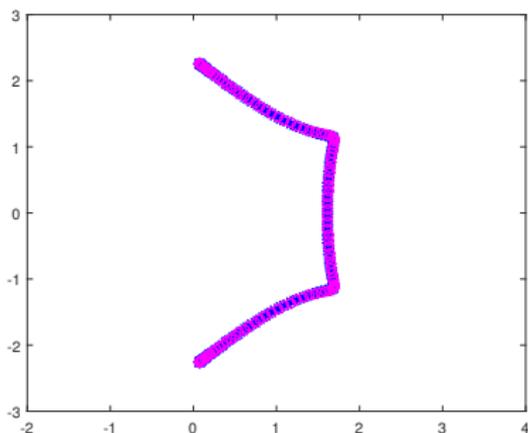
The points λ on the upper part of the asymptotic spectrum of G_n are given by

$$\begin{aligned} \operatorname{Re}(\lambda) &= 1 - \frac{1}{|\tilde{y}|^2} + \frac{\cos(\varphi)}{|\tilde{y}|} \left[1 - |\tilde{y}|^2 + \frac{2\cos(\varphi)}{|\tilde{y}|} + \frac{4\cos^2(\varphi) - 3}{|\tilde{y}|^2} \right], \\ \operatorname{Im}(\lambda) &= -\frac{\sin(\varphi)}{|\tilde{y}|} \left[1 + |\tilde{y}|^2 + \frac{2\cos(\varphi)}{|\tilde{y}|} + \frac{3 - 4\sin^2(\varphi)}{|\tilde{y}|^2} \right]. \end{aligned}$$

where r and s were defined above

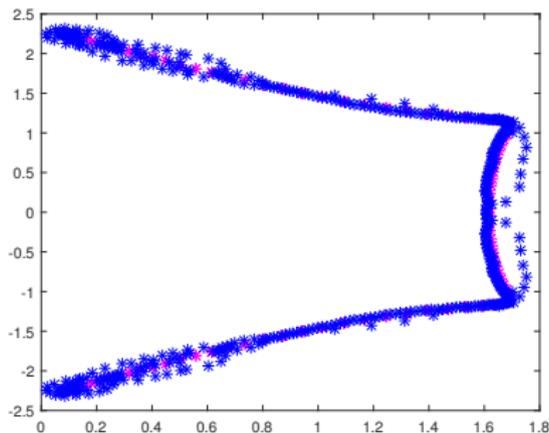


Upper part of the asymptotic spectrum, $\Delta(\zeta) > 0$ (blue) and $\Delta(\zeta) < 0$ (red)



QR eigenvalues (eig) of G_{200} (blue *) and asymptotic spectrum
(magenta *)

But, things are not always that nice...



QR eigenvalues (eig) of G_{500} (blue *) and asymptotic spectrum (magenta *)

The eigenvalues must be on or very close to the asymptotic spectrum. We see that the QR algorithm has difficulties computing accurate eigenvalues of G_n when n is large

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