The field of values of a bilinear form

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Mr. Toeplitz has recently shown in this journal[1] that the field of values W of a bilinear form, as a set of points in the complex plane, is bounded on the outside by a convex curve, but left open the question whether it fills the whole interior of this curve. It is now very easy to show that this question is to be answered in the affirmative, namely W is itself a convex set.

Let

$$C = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix}, \quad C^* = \begin{pmatrix} \bar{c}_{1,1} & \cdots & \bar{c}_{n,1} \\ \vdots & \ddots & \vdots \\ \bar{c}_{1,n} & \cdots & \bar{c}_{n,n} \end{pmatrix}$$

denotes an n-row matrix and its companion (conjugate-transpose) matrix, then two Hermitian matrices A, B, the components of C, are defined as

$$A = \frac{1}{2}(C + C^*), \quad B = \frac{1}{2i}(C - C^*),$$

or conversely

$$C = A + i\,B, \quad C^* = A - i\,B.$$

A linear substitution (non-vanishing determinant) with matrix P that transforms C into

$$\mathfrak{C} = PCP^*$$

simultaneously transforms C^* into $\mathfrak{C}^*$ and the components A, B into components $\mathfrak{A}$, $\mathfrak{B}$.

The field of values W of C is, according to Mr. Toeplitz, the set of the complex numbers w given by the values of the form

$$C(x, \bar{x}) = \sum_{\alpha,\beta} c_{\alpha,\beta}\, x_\alpha \bar{x}_\beta = w$$

when the complex variables $x_1, x_2, \ldots, x_n$ are subjected to the condition

$$(1) \quad E(x, \bar{x}) = \sum_{\alpha} x_\alpha \bar{x}_\alpha = 1.$$

[1] O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, Math. Zeitschrift 2 (1918), p. 187-197.

If we decompose it into its components, W is the set of real-valued pairs (u, v) or points with rectangular coordinates u, v with values given by a pair of Hermitian forms

$$A(x, \bar{x}) = u, \quad B(x, \bar{x}) = v$$

under condition (1).

By a unitary substitution ($PP^* = E$), which obviously does not change the field of values of C, we can transform the Hermitian matrix A into a diagonal matrix or assume

$$A(x, \bar{x}) = \sum_\alpha a_\alpha \, x_\alpha \bar{x}_\alpha$$

as a diagonal form with the eigenvalues $a_1, \ldots, a_n$. Let u_0 be a value that $A(x, \bar{x})$ assumes under the condition (1), which is between the smallest and the largest eigenvalue; geometrically speaking the straight line $u = u_0$, which is parallel to the v-axis, intersects the set W.

It is now simply a matter of showing that the set M of the value systems $x = (x_1, x_2, \ldots, x_n)$, for which

$$A(x, \bar{x}) = u_0, \quad E(x, \bar{x}) = 1$$

is valid, is contiguous. In fact, two such value systems or points x, y (in the space of n complex or $2n$ real dimensions) can be connected by a continuous curve running in M as follows, with $|x_\alpha| = \xi_\alpha$, $|y_\alpha| = \eta_\alpha$, let

$$x_\alpha = \xi_\alpha \, e^{i\,\phi_\alpha}, \quad y_\alpha = \eta_\alpha \, e^{i\,\psi_\alpha}$$

and for each of the three curves the parameter t runs through the interval $0 \leq t << 1$:

$$z_\alpha = \xi_\alpha \, e^{i\,(1-t)\phi_\alpha}$$

connects x to ξ,

$$z_\alpha = \sqrt{(1-t)\xi_\alpha^2 + t\eta_\alpha^2} \geq 0$$

connects ξ with η,

$$z_\alpha = \eta_\alpha \, e^{i\,t\psi_\alpha}$$

connects η with y.

The three arcs are

$$A(z, \bar{z}) = A(\xi, \xi) = A(\eta, \eta), \quad A(x, \bar{x}), \quad A(y, \bar{y}) = u_0$$

with also $E(z, \bar{z}) = 1$; the curve lies in M.

Moreover, M is bounded and closed. The values that $v = B(x, \bar{x})$ assumes as a continuous function of x in M, form a connected, bounded, closed set, i.e. an interval; the set W is intersected by every straight line $u = u_0$, in one line (possibly in a single point). Applied to the matrix $e^{i\,\phi}C$, whose field of values arises from W by a rotation, this says that the set W is intersected by any straight line of arbitrary direction in a segment: W is convex.

This procedure cannot be applied to more than two Hermitian forms

$$u_1 = A_1(x, \bar{x}),\ u_2 = A_2(x, \bar{x}),\ \ldots, u_r = A_r(x, \bar{x}).$$

The set W of points $(u_1, u_2, \ldots, u_r)$ which are obtained under the condition (1), is not necessarily convex. For $r = 3$ one can still prove like Mr. Toeplitz (op. cit. p. 194) on the basis of our two-dimensional result that W is intersected by every supporting plane in a convex plane set, i.e. is bounded externally by a convex surface; for $r > 3$ this is not generally true.

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