Chapter 4 Bases for Krylov subspaces

- 4.1 Orthonormal basis
- 4.2 Hessenberg basis
- 4.3 Biorthogonal basis
- 4.4 The generalized Hessenberg process
- 4.5 Q-OR optimal basis
- 4.6 Newton and Chebyshev bases
- 4.7 Truncated bases
- 4.8 Parallel computing

## 4.9 Finite precision arithmetic and numerical experiments



## 4.9.1 Orthogonal basis

**Fig. 4.1** fs 183 6,  $\log_{10} \|I - V_{n,k}^T V_{n,k}\|$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)



**Fig. 4.2** fs 183 6,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)

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4.9 Finite precision arithmetic and numerical experiments



Fig. 4.3 fs 183 6, numerical rank of  $V_{n,k}$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)



**Fig. 4.4** fs 183 6,  $\log_{10} ||I - V_{n,k}^T V_{n,k}||$ , Arnoldi MGS (plain), CGS (dashed), CGS with reorthogonalization (dot-dashed)

#### 4 Bases for Krylov subspaces



**Fig. 4.5** fs 183 6, MGS,  $\log_{10} \|I - V_{n,k}^T V_{n,k}\|$  (plain) and  $\log_{10}$  of  $cond(V_{n,k})$  (dashed)



**Fig. 4.6** fs 183 6, MGS,  $\log_{10} \|I - V_{n,k}^T V_{n,k}\|$  (plain) and  $\log_{10}$  of min<sub>y</sub>  $\|v_1 - AV_{n,k}y\|$ 



**Fig. 4.7** fs 680 1c,  $\log_{10} ||I - V_{n,k}^T V_{n,k}||$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)



**Fig. 4.8** fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)

#### 4 Bases for Krylov subspaces



Fig. 4.9 fs 680 1c, numerical rank of  $V_{n,k}$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)



**Fig. 4.10** random matrix n = 100,  $\log_{10} ||I - V_{n,k}^T V_{n,k}||$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)

4.9 Finite precision arithmetic and numerical experiments



**Fig. 4.11** random matrix n = 100, numerical rank of  $V_{n,k}$ , Arnoldi MGS (plain), CGS (dashed), Householder (dot-dashed)

## 4.9.2 Hessenberg basis



**Fig. 4.12** fs 183 6,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Hessenberg with pivoting (plain), without pivoting (dashed)



Fig. 4.13 fs 183 6, numerical rank of  $V_{n,k}$ , Hessenberg with pivoting (plain), without pivoting (dashed)

4.9 Finite precision arithmetic and numerical experiments



**Fig. 4.14** fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Hessenberg with pivoting (plain), without pivoting (dashed)

## 4.9.3 Biorthogonal basis



Fig. 4.15 fs 183 6,  $\log_{10}$  of the level of biorthogonality, biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)



**Fig. 4.16** fs 183 6,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)

4.9 Finite precision arithmetic and numerical experiments



Fig. 4.17 fs 183 6, numerical rank of  $V_{n,k}$ , biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)



Fig. 4.18 fs 680 1c,  $\log_{10}$  of the level of biorthogonality, biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)



**Fig. 4.19** fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)



Fig. 4.20 fs 680 1c, numerical rank of  $V_{n,k}$ , biortho (plain), biortho bis (dashed), biortho 2t (dot-dashed)

## 4.9.4 Q-OR opt basis



Fig. 4.21 fs 183 6,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Q-OR (plain), Q-OR t (dashed)



Fig. 4.22 fs 183 6, numerical rank of  $V_{n,k}$ , Q-OR (plain), Q-OR t (dashed)



Fig. 4.23 fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Q-OR (plain), Q-OR t (dashed)



Fig. 4.24 fs 680 1c, numerical rank of  $V_{n,k}$ , Q-OR (plain), Q-OR t (dashed)



Fig. 4.25 fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$  (plain) and lower bound (dashed)

## 4.9.5 Newton and Chebyshev bases



**Fig. 4.26** fs 183 6,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Arnoldi (plain), Newton RS (dashed), Chebyshev (dot-dashed)



**Fig. 4.27** fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , Arnoldi (plain), Newton RS (dashed), Chebyshev (dot-dashed)



## 4.9.6 Truncated Q-OR basis

**Fig. 4.28** fs 680 1c,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , different values of (p,q)



**Fig. 4.29** fs 680 1c, numerical rank of  $V_{n,k}$ , different values of (p, q)

4.9 Finite precision arithmetic and numerical experiments



**Fig. 4.30** supp 001,  $\log_{10}$  of minimum singular value of  $V_{n,k}$ , different values of (p, q)



Fig. 4.31 supg 001, numerical rank of  $V_{n,k}$ , different values of (p,q)

# Chapter 5 FOM/GMRES and variants

- 5.1 Introduction to FOM and GMRES
- 5.2 Convergence of FOM and GMRES
- 5.3 Prescribed convergence
- **5.4 Stagnation of GMRES**
- 5.5 Residual norms
- 5.6 The residual polynomials
- 5.7 Study of convergence using unitary matrices
- **5.8** Estimates of the norm of the error
- 5.9 Other implementations of GMRES
- 5.10 Parallel computing
- **5.11** Finite precision arithmetic

## 5.12 Numerical examples



## 5.12.1 FOM and GMRES

Fig. 5.1 fs 183 6,  $\log_{10}$  of the relative residual norm, GMRES (plain) and FOM (dashed)



Fig. 5.2 fs 183 6,  $\log_{10}$  of the relative true residual norm, GMRES (plain) and FOM (dashed)

#### 5.12 Numerical examples



Fig. 5.3 fs 183 6,  $\log_{10}$  of the relative error norm, GMRES (plain) and FOM (dashed)



Fig. 5.4 fs  $680 \text{ 1c}, \log_{10} \text{ of the relative residual norm, GMRES (plain) and FOM (dashed)}$ 



Fig. 5.5 e05r0500,  $\log_{10}$  of the relative residual norm, GMRES (plain) and FOM (dashed)

## 5.12.2 Variants of GMRES



Fig. 5.6 fs 183 6,  $\log_{10}$  of the relative true residual norm, GMRES, MGS (plain), CGS (dashed) and Householder (dot-dashed)



Fig. 5.7 fs 183 6,  $\log_{10}$  of the relative true residual norm, GMRES-MGS (plain) and level of orthogonality(dashed)



Fig. 5.8 fs 183 6,  $\log_{10}$  of the relative true residual norm, GMRES-MGS double-precision (plain) and GMRES-MGS variable precision with 64 decimal digits (dashed)



Fig. 5.9 fs 680 1c,  $\log_{10}$  of the relative true residual norm, GMRES, MGS (plain), CGS (dashed) and Householder (dot-dashed)

#### 5 FOM/GMRES and variants



Fig. 5.10 fs 680 1c,  $\log_{10}$  of the relative true residual norm, GMRES-MGS (plain) and level of orthogonality(dashed)



**Fig. 5.11** fs 680 1c,  $\log_{10}$  of the relative true residual norm, GMRES-MGS double-precision (plain) and GMRES-MGS variable precision with 64 decimal digits (dashed)

## 5.12.3 Prescribed convergence and stagnation



Fig. 5.12 Eigenvalues of A and Ritz values



Fig. 5.13 Relative difference between the prescribed values  $g_k$  and the GMRES true generated residual norms

### 5 FOM/GMRES and variants



Fig. 5.14 GMRES-MGS true residual norms

## 5.12.4 Residual norm bounds



Fig. 5.15 Example from  $\cite{eq: 1.1}$  , GMRES-MGS true residual norms and bound  $\cite{eq: 1.1}$ 



Fig. 5.16 fs 680 1c, GMRES-MGS true residual norms and bound

### 5 FOM/GMRES and variants



Fig. 5.17 supg 001, n = 1225, GMRES-MGS true residual norms and bound

## 5.12.5 Error norm estimates



Fig. 5.18 fs 183 6, GMRES-MGS, exact error norm (plain) and estimate, d = 1 (dashed)



Fig. 5.19 fs 680 1c, GMRES-MGS, exact error norm (plain) and estimate, d = 1 (dashed)

### 5 FOM/GMRES and variants



Fig. 5.20 e05r0500, GMRES-MGS, exact error norm (plain) and estimate, d = 1 (dashed)

# Chapter 6 Methods equivalent to FOM or GMRES

- 6.1 GCR, Orthomin, Orthodir and Axelsson's method
- 6.2 Simpler, residual-based simpler and adaptive simpler GMRES
- 6.3 Orthores
- 6.4 Q-OR Optinv
- 6.5 Parallel computing
- 6.6 Finite precision arithmetic

## 6.7 Numerical examples

## 6.7.1 GCR, Orthodir and Axelsson



Fig. 6.1 fs 183 6, true residual norms, GMRES-MGS (plain), GCR (dashed), Orthodir (dot-dashed), Axelsson (+)



Fig. 6.2 fs 680 1c, true residual norms, GMRES-MGS (plain), GCR (dashed), Orthodir (dot-dashed), Axelsson (+)

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Fig. 6.3 supg 001, true residual norms, GMRES-MGS (plain), GCR (dashed), Orthodir (dot-dashed), Axelsson (+)  $\,$ 

### 6.7.2 ASGMRES



Fig. 6.4 fs 183 6, relative true residual norms, GMRES-MGS (plain), SGMRES (dashed), RB GMRES (dot-dashed), ASGMRES  $\nu = 0.8$  (+)



Fig. 6.5 fs 680 1c, relative true residual norms, GMRES-MGS (plain), SGMRES (dashed), RB GMRES (dot-dashed), ASGMRES  $\nu = 0.8$  (+)



Fig. 6.6 supg 001, relative true residual norms, GMRES-MGS (plain), SGMRES (dashed), RB GMRES (dot-dashed), ASGMRES  $\nu = 0.8$  (+)



## 6.7.3 FOM and Orthores

Fig. 6.7 fs 183 6, relative true residual norms, FOM-MGS (plain), Orthores (dashed)



Fig. 6.8 fs 680 1c, relative true residual norms, FOM-MGS (plain), Orthores (dashed)
#### 6.7 Numerical examples



 $Fig. \, 6.9 \hspace{0.1 cm} \text{supg} \hspace{0.1 cm} \texttt{001}, order \hspace{0.1 cm} \texttt{1225}, relative true residual norms, FOM-MGS \hspace{0.1 cm} (plain), Orthores \hspace{0.1 cm} (dashed)$ 

### 6.7.4 Q-OR Optinv



Fig. 6.10 fs 183 6, relative true residual norms, GMRES-MGS (plain), Q-OR Optinv (dashed)



Fig. 6.11 fs 680 1c, relative true residual norms, GMRES-MGS (plain), Q-OR Optinv (dashed)



Fig. 6.12 supg 001, order 1225, relative true residual norms, GMRES-MGS (plain), Q-OR Optinv (dashed)

# Chapter 7 Hessenberg/CMRH

- 7.1 Derivation of the methods
- 7.2 Comparison with GMRES and convergence of CMRH
- 7.3 Prescribed convergence
- 7.4 Stagnation of CMRH
- 7.5 Residual norms
- 7.6 Parallel computing
- 7.7 Finite precision arithmetic



## 7.8 Numerical experiments

Fig. 7.1 fs 183 6, relative true residual norms, GMRES-MGS (plain), CMRH (dashed)



Fig. 7.2 fs 680 1c, relative true residual norms, GMRES-MGS (plain), CMRH (dashed)



Fig. 7.3 supg 001, order 1225, relative true residual norms, GMRES-MGS (plain), CMRH (dashed)

## Chapter 8 BiCG/QMR and Lanczos algorithms

- 8.1 The Lanczos algorithms
- 8.2 Derivation of BiCG
- 8.3 QMR
- 8.4 Breakdowns and look-ahead in BiCG/QMR
- 8.5 Comparison of QMR and GMRES
- 8.6 Prescribed convergence in BiCG and QMR
- 8.7 Residual norms
- 8.8 Lanczos algorithms with look-ahead and formal orthogonal polynomials
- 8.9 Parallel computing
- 8.10 Finite precision arithmetic

### 8.11 Numerical examples

#### 8.11.1 BiCG



Fig. 8.1 fs 183 6, relative true residual norms, BiCG (plain), Lanczos-Orthodir (dashed), Lanczos-Orthores (dot-dashed)



**Fig. 8.2** fs 183 6,  $\log_{10}$  of the relative true residual norm, BiCG (plain), BiCG variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 8.3 fs 680 1c, relative true residual norms, BiCG (plain), Lanczos-Orthodir (dashed), Lanczos-Orthores (dot-dashed)



**Fig. 8.4** fs 680 1c,  $\log_{10}$  of the relative true residual norm, BiCG (plain), BiCG variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 8.5 supg 001, order 1225, relative true residual norms, BiCG (plain), Lanczos-Orthodir (dashed), Lanczos-Orthores (dot-dashed)

#### 8.11.2 QMR



Fig. 8.6 fs 183 6, relative true residual norms, BiCG (plain), QMR 2t (dashed), QMR 3t (dot-dashed)



Fig. 8.7 fs 680 1c, relative true residual norms, BiCG (plain), QMR 2t (dashed), QMR 3t (dot-dashed)



Fig. 8.8 supg 001, order 1225, relative true residual norms, BiCG (plain), QMR 2t (dashed), QMR 3t (dot-dashed)

#### 8.11.3 Comparisons with GMRES



Fig. 8.9 raefsky1, relative residual norms versus iteration number, GMRES-MGS (plain), BiCG (dashed), QMR 2t (dot-dashed)



Fig. 8.10 raefsky1, relative residual norms versus number of matrix-vector products, GMRES-MGS (plain), BiCG (dashed), QMR 2t (dot-dashed)



 $\label{eq:Fig.8.11} \ \texttt{raefsky1}, \texttt{relative} \ \texttt{residual} \ \texttt{norms} \ \texttt{versus} \ \texttt{computing} \ \texttt{time}, \ \texttt{GMRES-MGS} \ \texttt{(plain)}, \ \texttt{BiCG} \ \texttt{(dashed)}, \ \texttt{QMR} \ \texttt{2t} \ \texttt{(dot-dashed)}$ 

8.11 Numerical examples

#### 8.11.4 Methods with look-ahead



 $Fig. \ 8.12 \ \ Relative \ true \ residual \ norms, \ GMRES-MGS \ (plain), \ HMRZ-stab \ (dashed)$ 

8.11.5 Ai/Bj

# Chapter 9 Transpose-free Lanczos methods

9.1 CGS

- 9.2 BiCGStab and extensions
- 9.3 Other product-type methods
- 9.4 Look-ahead for transpose-free methods
- 9.5 Parallel computing
- 9.6 Finite precision arithmetic

## 9.7 Numerical experiments

### 9.7.1 BiCGS



Fig. 9.1 fs 183 6, relative true residual norms, BiCG (plain), BiCGS (dashed)



Fig. 9.2 fs 680 1c, relative true residual norms, BiCG (plain), BiCGS (dashed)

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 $Fig. \ 9.3 \ \text{ supg } \ 001, \ \text{order } 1225, \ \text{relative true residual norms}, \ BiCG \ (plain), \ BiCGS \ (dashed)$ 



#### 9.7.2 BiCGStab and variants

Fig. 9.4 fs 183 6, relative true residual norms, BiCG (plain), BiCGStab (dashed)



**Fig. 9.5** fs 183 6,  $\log_{10}$  of the relative true residual norm, BiCGStab (plain), BiCGStab variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 9.6 fs 680 1c, relative true residual norms, BiCG (plain), BiCGStab (dashed)



**Fig. 9.7** fs 680 1c, log<sub>10</sub> of the relative true residual norm, BiCGStab (plain), BiCGStab variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)

#### 9 Transpose-free Lanczos methods



 $Fig. \ 9.8 \ \ \text{supg} \ \ 001, \ order \ 1225, \ relative \ true \ residual \ norms, \ BiCG \ (plain), \ BiCGStab \ (dashed)$ 



Fig. 9.9 fs 183 6, relative true residual norms, BiCGStab (plain), BiCGStab2 (dashed), BiCGtab( $\ell$ )  $\ell$  = 2 : 2 : 10



Fig. 9.10 fs 680 1c, relative true residual norms, BiCGStab (plain), BiCGStab2 (dashed), BiCGtab( $\ell$ )  $\ell$  = 2 : 2 : 10



Fig. 9.11 supg 001, order 1225, relative true residual norms, BiCGStab (plain), BiCGStab2 (dashed), BiCGtab( $\ell$ )  $\ell$  = 2 : 2 : 10



Fig. 9.12 raefsky1, relative true residual norms, BiCGStab (plain), BiCGStab2 (dashed), BiCGtab( $\ell$ )  $\ell$  = 2 : 2 : 10



Fig. 9.13 raefsky1, relative residual norms, BiCGStab (plain), BiCGStab2 (dashed), BiCGtab( $\ell$ )  $\ell$  = 2 : 2 : 10



Fig. 9.14 fs 680 1c, relative true residual norms, different BiCG-like algorithms



Fig. 9.15 fs 680 1c, relative true residual norms, BiCGStab (plain), CA-BiCGStab (dashed), I-BiCGtab (dot-dashed), K-BiCGStab (+), P-BiCGStab (o)

#### 9 Transpose-free Lanczos methods



Fig. 9.16 fs 680 1c, relative true residual norms, BiCGStab (plain), CA-BiCGStab s = 2 (dashed), s = 4 (dot-dashed), s = 8 (+), s = 16 (o)



Fig. 9.17 supg 001, order 1225, relative true residual norms, BiCGStab (plain), CA-BiCGStab (dashed), I-BiCGtab (dot-dashed), K-BiCGStab (+), P-BiCGStab (o)



 $\label{eq:Fig. 9.18} Fig. 9.18 \ raefsky1, \ relative \ true \ residual \ norms, \ BiCGStab \ (plain), \ CA-BiCGStab \ (dashed), \ I-BiCGstab \ (dot-dashed), \ K-BiCGStab \ (+), \ P-BiCGStab \ (o)$ 

#### 9.7.3 QMR-like methods



Fig. 9.19 fs 183 6, relative true residual norms, QMR 2t (plain), TFQMR (dashed)



Fig. 9.20 fs 680 1c, relative true residual norms, QMR 2t (plain), TFQMR (dashed)



Fig. 9.21 fs 183 6, relative true residual norms, different QMR-like algorithms



Fig. 9.22 fs 680 1c, relative true residual norms, different QMR-like algorithms

Chapter 10 The IDR family

- **10.1** The primitive IDR algorithm
- **10.2** The first IDR algorithm
- **10.3 IDR**(*s*) and variants
- **10.4 Other IDR algorithms**
- 10.5 Using higher order stabilizing polynomials
- 10.6 Residual norms and convergence
- 10.7 A predecessor of IDR: ML(k)BiCGStab
- **10.8 Parallel computing**
- **10.9** Finite precision arithmetic

## **10.10** Numerical examples





Fig. 10.1 fs 183 6, relative residual norms, different IDR algorithms



Fig. 10.2 fs 183 6, relative true residual norms, different IDR algorithms

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Fig. 10.3 fs 183 6,  $\log_{10}$  of the relative true residual norm, IDR-Bio(4) (plain), IDR-Bio(4) variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 10.4 fs 183 6,  $\log_{10}$  of the relative true residual norm, IDR-Bio(8) (plain), IDR-Bio(8) variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 10.5 fs 680 1c, relative residual norms, different IDR algorithms



Fig. 10.6 fs 680 1c, relative true residual norms, different IDR algorithms


**Fig. 10.7** fs 680 1c,  $\log_{10}$  of the relative true residual norm, IDR-Bio(4) (plain), IDR-Bio(4) variable precision with 32 decimal digits (dashed) and with 64 decimal digits (dot-dashed)



Fig. 10.8 raefsky1, relative true residual norms, different IDR algorithms



Fig. 10.9 raefsky1, relative residual norms, different IDR algorithms, residual norms vs time

## 10.10.2 Variants of IDR and IDR-QMR



Fig. 10.10 fs 680 1c, relative true residual norms, different Q-OR IDR partial orthogonalization algorithms



Fig. 10.11 fs 680 1c, relative true residual norms, different Q-MR IDR partial orthogonalization algorithms



Fig. 10.12 fs 680 1c, relative true residual norms, Q-MR IDR partial orthogonalization mnIDR algorithms, different values of s



Fig. 10.13 raefsky1, relative true residual norms, Q-MR IDR partial orthogonalization mnIDR algorithms, different values of s



Fig. 10.14 fs 680 1c, relative true residual norms, Q-MR IDR algorithms from [?], different values of s



Fig. 10.15 raefsky1, relative true residual norms, Q-MR IDR algorithms from [?], different values of s



Fig. 10.16 fs 680 1c, relative true residual norms, IDRStab algorithms from [?], different values of s and  $\ell$ 



Fig. 10.17 raefsky1, relative true residual norms, IDRStab algorithms from [?], different values of s and  $\ell$ 

### 10.10.3 ML(k)BiCGStab



Fig. 10.18 fs 680 1c, relative true residual norms, IDR Bio(s) and ML(k)BiCGStab algorithms, different values of s and k



Fig. 10.19 raefsky1, relative true residual norms, IDR Bio(s) and ML(k)BiCGStab algorithms, different values of s and k

# Chapter 11 Restart, deflation and truncation

- **11.1 Restarted FOM and GMRES**
- **11.2 Prescribed convergence**
- 11.3 Restarting techniques for FOM and GMRES
- 11.4 Restarted and augmented CMRH and Q-OR Opt
- **11.5 Truncated FOM and GMRES**
- 11.6 Truncated Q-OR
- **11.7 Parallel computing**
- **11.8 Finite precision arithmetic**

# **11.9** Numerical experiments



# 11.9.1 Restarted GMRES without preconditioning

Fig. 11.1 fs 183 6, relative true residual norms, GMRES-MGS(m), different values of m



Fig. 11.2 fs 680 1c, relative true residual norms, GMRES-MGS(m), different values of m

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#### 11.9 Numerical experiments



Fig. 11.3 supg 001, relative true residual norms, GMRES-MGS(m), different values of m



Fig. 11.4 diff conv 400, relative true residual norms, GMRES-MGS(m), different values of m



Fig. 11.5 raefsky1, relative true residual norms, GMRES-MGS(m), different values of m



Fig. 11.6 raefsky1, relative residual norms vs time, GMRES-MGS(m), different values of m

## 11.9.2 Restarted GMRES with preconditioning



Fig. 11.7 supp 001 22500, relative true residual norms, GMRES-MGS(m) with SSOR preconditioning, different values of m



Fig. 11.8 supp 001 22500, relative true residual norms vs time, GMRES-MGS(m) with SSOR preconditioning, different values of m



Fig. 11.9 supp 001 22500, relative true residual norms, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.10 supp 001 22500, relative true residual norms vs time, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.11 rajat27b, relative true residual norms, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.12 rajat27b, relative true residual norms vs time, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.13 matrix-new 3, relative true residual norms, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.14 matrix-new 3, relative true residual norms vs time, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



Fig. 11.15 matrix-new 3, relative true residual norms vs time, GMRES-MGS(m) with ILU(0) preconditioning, different values of m



# **11.9.3** Restarting with deflation and augmentation without preconditioning

Fig. 11.16 fs 680 1c, relative true residual norms, full GMRES and GMRES(m) with different restarting techniques



Fig. 11.17 fs 680 1c, relative true residual norms, full GMRES, GMRES(40) and GMRES-DR(40) with different number of harmonic Ritz vectors



Fig. 11.18 fs 680 1c, relative true residual norms, full GMRES, GMRES(40) and GMRES-DR(m) with different number of harmonic Ritz vectors



Fig. 11.19 fs 680 1c, eigenvalues and harmonic Ritz values, iterations 20:5:40



Fig. 11.20 fs 680 1c, relative true residual norms, GMRES(35), GMRES-DR(35,5), LGM-RES(30,5) and HBGMRES(34)



Fig. 11.21 fs 680 1c, relative true residual norms, GMRES(40), GMRES-DR(40,5), LGM-RES(35,5) and HBGMRES(39)

## 11.9 Numerical experiments



**Fig. 11.22** supg 001, n = 1225, relative true residual norms, full GMRES and GMRES(m) with different restarting techniques



Fig. 11.23 supg001, n = 1225, relative true residual norms, full GMRES, GMRES(40) and GMRES-DR(40) with different number of harmonic Ritz vectors



**Fig. 11.24** supg001, n = 1225, relative true residual norms, full GMRES, GMRES(40) and GMRES-DR(m) with different number of harmonic Ritz vectors



Fig. 11.25 supg 001, n = 1225, eigenvalues and harmonic Ritz values, iterations 20:5:40



**Fig. 11.26** supg 001, n = 1225, relative true residual norms, GMRES(35), GMRES-DR(35,5), LGMRES(30,5) and HBGMRES(34)



Fig. 11.27 supp 001, n = 1225, relative true residual norms, GMRES(35), GMRES-DR(40,5), LGMRES(35,5), HBGMRES(39) and GMRES(40)

# **11.9.4** Restarting with deflation and augmentation with preconditioning



# 11.9.5 Truncated Q-OR

**Fig. 11.28** fs 680 1c, relative true residual norms, full GMRES, QOR-T(1,40), QOR-T(20,20) and QOR-T(20,20) with restart at k = 120



**Fig. 11.29** supp 001, n = 1225, relative true residual norms, full GMRES, QOR-T(1,20), QOR-T(10,10) and QOR-T(10,10) with restart at k = 100

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Chapter 12

12 Related topics

# **Related topics**

- **12.1 FGMRES**
- 12.2 GCRO
- 12.3 Relaxation for matrix-vector products
- 12.4 Systems with multiple right-hand sides
- 12.5 Shifted linear systems
- **12.6 Singular systems**
- 12.7 Least squares problems
- **12.8 Ill-posed problems**
- 12.9 Eigenvalue computations and rational Krylov methods
- **12.10** Functions of matrices
- 12.11 Minimum error methods
- 12.12 Residual replacement techniques
- 12.13 Residual smoothing techniques
- 12.14 Hybrid methods
- 12.15 CGNE and CGNR
- 12.16 USYMLQ and USYMQR

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# Chapter 13 Numerical comparisons of methods

# **13.1 Introduction**

# **13.2 Small matrices**



Fig. 13.1 Small matrices, minimum number of matvecs,  $\epsilon = 10^{-6}$ 

### 13 Numerical comparisons of methods



Fig. 13.2 Small matrices, minimum computing time,  $\epsilon = 10^{-6}$ 



Fig. 13.3 Small matrices, minimum number of matvecs,  $\epsilon = 10^{-10}$ 



Fig. 13.4 Small matrices, minimum computing time,  $\epsilon = 10^{-10}$ 

# 13.3 Larger matrices



Fig. 13.5 supg 001 22500, relative true residual norms vs matvecs, ILU(0) preconditioning



Fig. 13.6 supg 001 22500, relative true residual norms vs time, ILU(0) preconditioning

### 13 Numerical comparisons of methods



Fig. 13.7 supg 001 22500, relative true residual norms vs time, zoom, ILU(0) preconditioning



Fig. 13.8 rajat27b, relative true residual norms vs matvecs, ILU(0) preconditioning



Fig. 13.9 rajat27b, relative true residual norms vs time, ILU(0) preconditioning

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### 13.3 Larger matrices



Fig. 13.10 rajat27b, relative true residual norms vs time, zoom, ILU(0) preconditioning



Fig. 13.11 matrix-new 3, relative true residual norms vs matvecs, ILU(0) preconditioning



Fig. 13.12 matrix-new 3, relative true residual norms vs time, ILU(0) preconditioning

### 13 Numerical comparisons of methods



Fig. 13.13 matrix-new 3, relative true residual norms vs time, zoom, ILU(0) preconditioning



Fig. 13.14 Large matrices, minimum number of matvecs,  $\epsilon = 10^{-6}$ 



Fig. 13.15 Large matrices, minimum computing time,  $\epsilon = 10^{-6}$ 

### 13.3 Larger matrices



Fig. 13.16 Large matrices, minimum number of matvecs,  $\epsilon = 10^{-10}$ 



Fig. 13.17 Large matrices, minimum computing time,  $\epsilon = 10^{-10}$