

The approximate determination of harmonic functions and conformal mappings

Heinrich Liebmann

Bayerischen Akad. Wiss. München, v 47, (1918), pp. 385-416

This is a partial translation of Liebmann's paper.

§1. Boltzmann's "proof" of Dirichlet's principle.

1. Boltzmann's approach and its significance.

Ludwig Boltzmann, working in Munich in the winter semester 1892/1893, gave a lecture on the mechanical potential, which has been published after the tragic end of the researcher, by his student, the astronomer Hugo Buchholz¹. The editor sees a highlight in the treatment that Boltzmann gave to Dirichlet's principle; it sees here a "new proof" of the principle and calls for the "pure" mathematician to treat the question further in this sense.

It seems that this call has faded away unheard; indeed, it is reported that authoritative researchers consider the path taken to be quite unsuitable for arriving at a new rigorous proof. We will comment on this below (§2, 2), but we may refer to the fact that in view of the proofs given by C. Neumann, H. A. Schwarz, Poincaré and Hilbert, a new proof is hardly felt to be necessary. We will leave it open whether in this sense - "nothing at all comes out of it", but we are inclined to a milder view.

In any case, Boltzmann's idea - and therein lies its undeniable lasting value - can be extended to a practical approximate solution of the boundary value problem, which leaves nothing to be desired in terms of simplicity and deserves to be put into the service of applied mathematics, together with the method of Runge².

¹Hugo Buchholz, *The mechanical potential according to lectures of L. Boltzmann and the theory of the figure of the earth*. Leipzig 1908. (In particular p. VIII and ch. VIII, pp. 150-176.)

²The area of numerical and graphic integration is presented in detail in Article II C 2 of *math. Encyclopedia* (Runge and Willers). Above all, those mentioned in Chap. III (graphic and numerical integration of partial differential equations) discussed works by Runge himself (footnote 529, p. 165, 549, p. 173) and his student Rottsieper, who has since fallen into the field of honor. Finally, we mention also the extensive work of Richardson (footnote 548, p. 173). We will come back to the latter: *Approximate arithmetical solution by finite differences of physical problems* (London R. S. Philos. Trans. A 210 (1910), p. 507-

Boltzmann replaces the differential equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

by the difference equation

$$D_2 u \equiv u(x - \varepsilon, y) + u(x + \varepsilon, y) + u(x, y - \varepsilon) + u(x, y + \varepsilon) - 4u(x, y) = 0$$

and he can then solve the (first) boundary value problem first approximated for a square in the following way: The square is decomposed by equidistant parallels to the sides into a mesh of n^2 squares, so that a grid of $(n - 1)^2$ inner points and $4(n - 1)$ edge points is created - the four corner points do not matter at first. For the determination of the values $u_{i,k}$ of the solution of the difference equation then the $(n - 1)^2$ equations serve, which express the requirement that - according to the difference equation - $u_{i,k}$ should be equal to the arithmetic mean of the values in the four neighboring grid points; the $4(n - 1)$ values at the edge are given. One then obtains the $u_{i,k}$ by solving these equations.

Whether the limit for $n = \infty$ can be carried out directly, our main question is: How can the difference equation be solved?

2. A series of questions. Following the suggestion of Boltzmann (resp. Buchholz), we have to discuss a number of points, which we list here separately.

1) Direct determination of the values of the grid function by solving the difference equation for $n = 2, 3, 4, \dots$ - As expected, a limit soon appears, at which the computation becomes impossible.

2) Approximate solution of the difference equation and proof of convergence towards the rigorous solution. An idea occasionally used by Jacobi gave the opportunity for this³; it was the remainder of these considerations which made the practical realization of Boltzmann's idea possible.

3) Change of the area : Instead of the square, other simply connected areas of the plane can occur, which are bounded, for example, by a polygonal line or, more generally, by a curve.

4) Replacement of the rectangular parallel coordinate mesh by other isothermal coordinates adapted to the boundary shape. In particular, the mesh of isothermally divided polar coordinates used by Runge with great success comes into consideration.

5) Application to the conformal mapping of the interior of a simply connected area to the interior of the circle.

537) (§3, 3), because, in spite of entirely different methods, of all the above, it shows the closest relation with our explanations.

³Jacobi explained how to solve linear equations, whose immediate solution is inconvenient, can be done with an iterative approximation process (Ges. Werke III (Berlin 1884), pp. 467-478). In the first place, Jacobi thinks of the normal equations of the method of least squares.

If - by a peculiar method closely connected with the now so called “distortion theorems” - this mapping is once done, then (what also was used by Runge) the boundary value problem for this region is led back to the boundary value problem for the circle. We will go further and show how a simple area of the plane, bounded by an outer and an inner boundary curve, can be mapped to the ring area between two concentric circles. This task, in which, as is well known, only the radius of one circle can be chosen arbitrarily, is also of outstanding practical importance for determining the lift of biplanes⁴.

6) Approach for the solution of the boundary value problem in space, to which Buchholz (and later Richardson) has pointed out. With due patience, the “unsolved Dirichlet problem”, the solution of the boundary value problem for the cube, becomes accessible to calculation. Of course, closed expressions composed of known functions and operational symbols cannot be reached from the difference equations. But, for example, the temperature distribution in the interior of a mountain with given surface values, a problem often treated in the literature on tunnel construction, can be attacked in this way. Incidentally: a tabular solution, that was originally one of the end purposes of analytical investigations, and where the tracing back to “known functions” - that is, known tables - is not successful, applied mathematics comes in, which in any case paves the way to immediate tabulation.

The most suitable name for the method to be described would be: *Method of arithmetic mean*. But since it has already been used in a different meaning in the fundamental research of C. Neumann, we choose the word *determination* or *computation*, which perfectly reflects the course of the calculation.

§2. The direct solution of the mesh problem.

1. The boundary value problem for the square, the rectangle and the cube.

A first quite crude approximation for the solution of the boundary value problem for the square, at the same time the exact solution of the difference equation, or, as we want to say more clearly, of the mesh problem for $n = 2$ for the square, would be that one assigns to the center of the square, as function value, the arithmetic mean of the four values prescribed for the centers of the side neighbours.

We want to report the results for $n = 4$ (nine points inside) and $n = 6$ (25 points inside), partly in the form of a table of the weights, partly in the form of a table of the coefficients of the boundary values.

⁴The Kutta-Joukowski theory of the lift of a wing profile (reports 1910 and 1911) requires the determination of conformal mappings. See also v. Mises, Z.f. Flugtechnik und Motorluftschiffahrt 1917, p. 157 ff. - See also §4, 3 below.

The following tables give the weights for $n = 4$

$$\begin{array}{ccccccccc}
 & 1 & 2 & 1 & & 11 & 37 & 11 & \\
 1 & \cdot & \cdot & \cdot & 1 & 11 & \cdot & u_{1,2} & \cdot & 11 \\
 2 & \cdot & u_{2,2} & \cdot & 2 & 7 & \cdot & \cdot & \cdot & 7 \\
 1 & \cdot & \cdot & \cdot & 1 & 3 & \cdot & \cdot & \cdot & 3 \\
 & 1 & 2 & 1 & & 3 & 5 & 3 & & \\
 [p] & = & 16 & & & [p] & = & 112 & &
 \end{array}$$

$$\begin{array}{cccc}
 & 67 & 22 & 7 \\
 67 & u_{1,1} & \cdot & \cdot & 7 \\
 22 & \cdot & \cdot & \cdot & 6 \\
 7 & \cdot & \cdot & \cdot & 3 \\
 & 7 & 6 & 3 & \\
 [p] & = & 224 & &
 \end{array}$$

[Note from the translator: These coefficients, divided by p , give the entries of a row of the matrix inverse, except for the center point (the 5th one) but it will be multiplied by 0 for a zero right-hand side of the PDE]

They express the weights to be applied to the 12 values at the edge to calculate the relevant value of the mesh function. In order to obtain the desired $u_{i,k}$, one has to form the sum of the function values in the edge points of the grid, each multiplied by its weight, and then to divide it by the relevant weight sum $[p]$. The following simple example with the boundary values 1, 2, 3 and 4

[Note from the translator: The a 's in Liebmann's paper must be u 's]

$$\begin{array}{cccccc}
 u_{0,1} = 1 & & u_{0,2} = 1 & & u_{0,3} = 1 & \\
 \\
 u_{1,0} = 4 & u_{1,1} = \frac{35}{14} = 2.50 & u_{1,2} = \frac{109}{56} = 1.95 & u_{1,3} = \frac{25}{14} = 1.79 & u_{1,4} = 2 & \\
 u_{2,0} = 4 & u_{2,1} = \frac{171}{56} = 3.05 & u_{2,2} = \frac{5}{2} = 2.50 & u_{2,3} = \frac{123}{56} = 2.20 & u_{2,4} = 2 & \\
 u_{3,0} = 4 & u_{3,1} = \frac{45}{14} = 3.21 & u_{3,2} = \frac{157}{56} = 2.80 & u_{3,3} = \frac{35}{14} = 2.50 & u_{3,4} = 2 & \\
 \\
 u_{4,1} = 3 & & u_{4,2} = 3 & & u_{4,3} = 3 &
 \end{array}$$

will be easily understandable, even if the first intermediate calculations are omitted.

For $n = 6$, i.e. for 25 inner mesh points, the weighted sum - after of course the weights are chosen in such a way that they have no more common divisor

- in the worst case $u_{1,2}$ already rises to 102960.

$$\begin{array}{ccccc}
 & 3 & 6 & 8 & 6 & 3 \\
 3 & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 8 & \cdot & \cdot & u_{3,3} & \cdot & \cdot & 8 \\
 6 & \cdot & \cdot & \cdot & \cdot & \cdot & 6 \\
 3 & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\
 & 3 & 6 & 8 & 6 & 3 \\
 & [p] & = & 104 & & &
 \end{array}$$

It may therefore suffice to give the weight table only for the center ($u_{3,3}$), but only the coefficients for the rest, i.e. the weights divided by the weight sum $[p]$ to four decimals. The coefficients given because of the symmetry do not need to be written again. We write them in the sequence that we first start with the coefficient of $u_{0,1}$ and then go around the boundary of the square in the clockwise way.

For $u_{1,1}$:

$$\begin{array}{l}
 \underline{0.3012}, \quad 0.1035, \quad 0.0402, \quad 0.0169, \quad 0.0064; \\
 0.0064, \quad 0.0088, \quad 0.0079, \quad 0.0055, \quad 0.0027.
 \end{array}$$

For $u_{2,2}$:

$$\begin{array}{l}
 0.0720, \quad \underline{0.1439}, \quad 0.0928, \quad 0.0492, \quad 0.0208; \\
 0.0208, \quad 0.0341, \quad 0.0322, \quad 0.0227, \quad 0.0114.
 \end{array}$$

For $u_{3,3}$:

$$0.02885, \quad 0.0577, \quad \underline{0.0769} \quad \text{and so on.}$$

In the coefficient tables for $u_{1,3}$ and $u_{2,3}$ we start with the coefficient of $u_{0,3}$ and end with that of $u_{5,3}$. It yields

For $u_{1,3}$:

$$\begin{array}{l}
 \underline{0.3483}, \quad 0.1203, \quad 0.0402; \\
 0.0402, \quad 0.0405, \quad 0.0288, \quad 0.0172, \quad 0.0079; \\
 0.0079, \quad 0.0143, \quad 0.0170.
 \end{array}$$

For $u_{2,3}$:

$$\begin{array}{l}
 \underline{0.1527}, \quad 0.0928, \quad 0.0405; \\
 0.0405, \quad 0.0690, \quad 0.0577, \quad 0.0367, \quad 0.0172; \\
 0.0172, \quad 0.0322, \quad 0.0396.
 \end{array}$$

The coefficient table for $u_{1,2}$ finally does not show any symmetry any more, therefore it is to be written down completely.

For $u_{1,2}$:

$$\begin{array}{l}
 0.1035, \quad \underline{0.3420}, \quad 0.1203, \quad 0.0466, \quad 0.0169; \\
 0.0169, \quad 0.0208, \quad 0.0172, \quad 0.0114, \quad 0.0055; \\
 0.0055, \quad 0.0106, \quad 0.0143, \quad 0.0143, \quad 0.0088; \\
 0.0088, \quad 0.0208, \quad 0.0404, \quad 0.0720, \quad 0.1035.
 \end{array}$$

The coefficient of the nearest boundary point to the $u_{i,k}$ in question has been underlined each time, it always has the largest value in the sequence; the coefficients naturally decrease with increasing distance from the point and the boundary point.

Whether one arrives at a law in this way may remain an open question; in any case, one soon arrives at a picture of the course which provides sufficient insight for many practical purposes.

In passing, some remarks about rectangles and cubes are allowed. For the rectangle with $3 \cdot 7 = 21$ inner mesh points, the weight sum already increases to $161,352,128!$ ⁵ The calculation was carried out by applying a merging procedure that shows a relationship with the methods of potential theory: Place two squares, each with nine interior mesh points, end to end so that the right side of the first coincides with the left side of the second, then express the nine values

$$u_{i,k} \quad (i = 1, 2, 3; k = 1, 2, 3)$$

and the nine values

$$u_{i,k} \quad (i = 1, 2, 3; k = 5, 6, 7)$$

by the prescribed boundary values on the left and right half of the rectangle and the unknown $u_{1,4}, u_{2,4}, u_{3,4}$, on the central vertical. If we then add the three equations that arise from the connector

$$\begin{array}{ccccc} & & u_{0,4} & & \\ & & u_{1,3} & u_{1,4} & u_{1,5} \\ & & u_{2,3} & u_{2,4} & u_{2,5} \\ & & u_{3,3} & u_{3,4} & u_{3,5} \\ & & & u_{4,4} & \end{array}$$

that is, the equations that can be read from the weight tables

$$\begin{array}{ccccccc} & & 15 & & 1 & & 1 \\ 15 & u_{1,4} & 15 & 1 & \cdot & 1 & 1 & \cdot & 1 \\ 4 & \cdot & 4 & 4 & u_{2,4} & 4 & 4 & \cdot & 4 \\ 1 & \cdot & 1 & 1 & \cdot & 1 & 15 & u_{3,4} & 15 \\ & 1 & & & 1 & & & 15 & \\ [p] & = & 56 & [p] & = & 14 & [p] & = & 56 \end{array}$$

and we can solve the 21 equations with the use of previous results.

In general, if you want to determine the coefficients of a figure created by merging two figures, one has to consider an unfortunate circumstance: It is difficult to specify the accuracy to be prescribed for the coefficients of

⁵Runge arrives at similar very large figures when it comes to integration of $\Delta u = \text{const}$ (see Math. and Phys. 56 (1908), pp. 225-233), where 43 unknowns were to be determined.

the partial figures and of the merging strip, which would be necessary to obtain, for the total figure, the coefficients accurate to a prescribed number of decimals!

Finally we want to give a result for the cube with 125 inner grid points, that is 25 points on each side face. In space one has to replace the differential equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

by the difference equation

$$\begin{aligned} D_2 u \equiv & u(x - \varepsilon, y, z) + u(x + \varepsilon, y, z) + u(x, y - \varepsilon, z) + u(x, y + \varepsilon, z) \\ & + u(x, y, z - \varepsilon) + u(x, y, z + \varepsilon) - 6u(x, y) = 0 \end{aligned}$$

and in the considered case, if one wants to calculate the value of the mesh function in the center, one obtains on each side face the same symmetrical weight table

$$\begin{array}{ccccc} 5 & 10 & 13 & 10 & 5 \\ 10 & 21 & 29 & 21 & 10 \\ 13 & 29 & 44 & 29 & 13 \\ 10 & 21 & 29 & 21 & 10 \\ 5 & 10 & 13 & 10 & 5 \end{array}$$

which again shows the everywhere recognizable qualitative distribution with a weight maximum at the point of the boundary, which is closest to the inner mesh point - here the center of the cube. The weight sum here is

$$6 \cdot \{44 + 4(29 + 21 + 13 + 5) + 8 \cdot 10\} = 2376.$$

2. The convergence of the mesh function against the harmonic function. We believe we must begin with the harsh judgment that a generalized investigation of the convergence of the mesh function against the sought harmonic function is of no value, as we are firmly convinced that this convergence can be obtained with increasing densification of the mesh and extraction of the values of the mesh function to be prescribed in the boundary points from the betrayal of values of the harmonic function prescribed at the edge and which is to be determined in the interior. Certainly, one can make an estimation by decomposing the mesh function as the harmonic one increased by a remainder whose one factor is the third power of the interval size ε , while the other is given by an aggregate of the mean values of the third partial derivatives provided with number coefficients⁶. But these third partial derivatives will only remain within finite bounds in the whole interior under very tight conditions! This serious concern against a uniform residual estimate can be partly softened by considering that with n^2 points inside (n

⁶See Richardson, op. cit., p. 310.

being a given number), one never comes near the critical point where the finiteness of the partial derivatives ends.

In any case, the convergence of the mesh function against the solution of the boundary value problem, which is assumed to exist, is difficult to prove by a method that does justice to all cases, and it seems all the worse with the attempt to prove the existence of the solution of the boundary value problem from the mesh function for limit $n = \infty$!

Things seem to be much favorable if, for example, the differential equation

$$\Delta u = k^2 u$$

and the associated difference equation

$$D_2 u \equiv u(x-\varepsilon, y) + u(x+\varepsilon, y) + u(x, y-\varepsilon) + u(x, y+\varepsilon) - 4u(x, y) = \varepsilon k^2 u(x, y)$$

are considered.

If for $\Delta u = 0$ and $\Delta u = k^2 u$, rather for the two associated difference equations, the progressive mesh division is carried out in such a way for ($n = 2, 4, 8, 16, \dots$) that each mesh point occurring in one mesh occurs again in all following ones, then it is shown that the values

$$u(x, y) \dots \left(x = \frac{\mu}{2^m}, y = \frac{\nu}{2^m}, \dots \mu, \nu, m \text{ given} \right)$$

solving the first difference equation with increasing division of the mesh form a (of course, convergent) non-monotonic sequence, while this sequence is monotonic from the start in the more general difference equation and in any case also becomes monotonic in other examples from a certain point on. If the monotonous character of the sequence can be demonstrated, the proof of convergence will certainly be easier to give. So all the more detailed circumstances seem to indicate that on the way via the difference equation the existence of the solution of the boundary value problem can be obtained first for the more general differential equation and then probably also by looking for the limit for $k = 0$. It would then turn out to be that Boltzmann's "determination method" repeat exactly what C. Neumann said for his method of arithmetic mean⁷: that it leads to the proof of existence more easily for k different from zero than for $k = 0$.

So much for the moment about the implementation of the idea of Boltzmann's "proof of existence".

3. An example. How far the agreement between the lattice function and the harmonic function can go "under favorable circumstances" is shown by the example $u(x, y) = \sin x \sinh y$. Calculated for the mesh points

$$x = \frac{k}{8}, y = \frac{\ell}{8}, \quad (k, \ell = 0, 1, \dots, 8),$$

⁷See Math. Enc. II A 7 c: Boundary value problems in the theory of partial differential equations (Sommerfeld), No. 11.

so the difference equation in the square is exactly valid up to three units of the fifth decimal place! We want to quote a few numbers for this example.

We have

$$\begin{aligned} u(0, \frac{1}{2}) &= 0, \quad u(\frac{1}{2}, 0) = 0, \quad u(1, 0) = 0, \quad u(0, 1) = 0, \\ u(1, \frac{1}{2}) &= 0.43848, \quad u(\frac{1}{2}, 1) = 0.56329, \quad u(1, 1) = 0.98889, \end{aligned}$$

and

$$u(\frac{1}{2}, \frac{1}{2}) = 0.24982.$$

The mean value

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} \left(u(0, \frac{1}{2}) + u(\frac{1}{2}, 0) + u(1, \frac{1}{2}) + u(\frac{1}{2}, 1) \right) = 0.25405$$

which is quite roughly formed, is already in very good agreement with this real value.

It is also possible to form the mean - by rotating the coordinate system by the angle $\pi/4$ - in such a way that one starts from the corners instead of from the side centers (diagonal mean) and then obtains

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} (u(0, 0) + u(0, 1) + u(1, 0) + u(1, 1)) = 0.24722.$$

One gets even closer to the correct value if one averages the approximate values with regard to their weights, which in any case are inversely proportional to the distances of the mesh point $(1/2, 1/2)$ from the “neighboring points” used for averaging (here very far away) in our case one obtains

$$\frac{\sqrt{2}f\left(\frac{1}{2}, \frac{1}{2}\right) + 1 \cdot \bar{f}\left(\frac{1}{2}, \frac{1}{2}\right)}{1 + \sqrt{2}} = 0.24980,$$

instead of 0.24982.

This example now suggests a train of thought that leads to the goal: Shouldn't one arrive at the calculation of the mesh function by first obtaining a value for the center of the square by brute averaging, then continuing the calculation using this value, and so on? And is it not possible to adjust a system of values of the function for n^2 mesh points, obtained in this or another way, afterwards, so that one comes indefinitely closer to the sought values, which, after all, could be obtained by direct resolution of n^2 equations only by way of a long and laborious calculation?

§3. The application of Jacobi's method.

1. The grid function for the square. This is the solution of the system of equations

$$4u_{i,k} = u_{i-1,k} + u_{i+1,k} + u_{i,k-1} + u_{i,k+1}, \quad (i, k = 1, 2, \dots, n-1),$$

where the boundary values

$$u_{0,k} = a_{0,k}, \quad u_{n,k} = a_{n,k}, \quad u_{i,0} = a_{i,0}, \quad u_{i,n} = a_{i,n},$$

are given numbers. The equations arise from the requirement that the sum of the squares of the differences

$$\sum \sum \left\{ (u_{i-1,k} - u_{i,k})^2 + (u_{i+1,k} - u_{i,k})^2 + (u_{i,k-1} - u_{i,k})^2 + (u_{i,k+1} - u_{i,k})^2 \right\}$$

over all inner points of the square should become as small as possible with given boundary values - this was the idea of Boltzmann, which was very obvious in itself. On the other hand, Jacobi has studied how to obtain an approximate solution of the "normal equations" to which the least squares method leads. It is actually obvious to apply Jacobi's method to Boltzmann's difference equations.

We describe the procedure for our purpose, a modification of Jacobi's method, and then prove its convergence, which is very easy to achieve.

If (for a given n) you first assign some values $u_{i,k}^0$ to the inner mesh points and then the values on the inner lines closest to the boundary with the help of the difference equation itself, i.e. the values $u_{i,k}^0$ for

$$(i = 1, k = 1, \dots, n-1), \quad (i = n-1, k = 1, \dots, n-1),$$

$$(i = 1, \dots, n-1, k = 1), \quad (i = n-1, \dots, n-1, k = 1),$$

are replaced by

$$u'_{i,k} = \frac{1}{4} \left(u_{i-1,k}^0 + u_{i+1,k}^0 + u_{i,k-1}^0 + u_{i,k+1}^0 \right)$$

- the boundary values are fixed - the mesh function of the first and last mesh lines is improved. With these improved values and the values on the third and $(n-3)$ th lines, one improves the values in the second and $(n-2)$ th lines and so on. When you have reached the innermost square, you start again from the outside and calculate to the center, and so on.

We claim that the values improved in this way converge in geometric progression towards the sought values.

We know that for given boundary values there is a single, well-defined mesh function which strictly satisfies the difference equation and is therefore no longer changed by the approximation method. Let this strict solution be denoted by U , or by $U_{i,k}$. If $v_{i,k}$ is an arbitrary iterate, but with correct boundary values, then it is to be shown that the method applied to

$$u_{i,k}^0 = U_{i,k} - v_{i,k}$$

with unlimited repetition makes the values of this mesh function converge to zero.

For the proof it is sufficient to show that, with zero uniformly prescribed boundary values, the improved values converge to zero. In this case, let $-M_1$ be the smallest negative and $+M_2$ the largest positive among the arbitrarily assumed values, then after the first improvement the values on the first inner ring of mesh points lie between the bounds

$$-\frac{3M_1}{4} \text{ and } +\frac{3M_2}{4}.$$

If you then improve on the second inner ring, you get the bounds

$$-\frac{1}{4} \left(\frac{3M_1}{4} + 3M_1 \right) = -M_1 \left(1 - \frac{1}{16} \right)$$

and

$$\frac{1}{4} \left(\frac{3M_2}{4} + 3M_2 \right) = M_2 \left(1 - \frac{1}{16} \right)$$

for the improved values.

In general, the bounds

$$-\frac{1}{4} \left(1 - \frac{1}{4^{k-1}} + 3 \right) M_1 = -M_1 \left(1 - \frac{1}{4^k} \right)$$

and

$$\frac{1}{4} \left(1 - \frac{1}{4^{k-1}} + 3 \right) M_2 = M_2 \left(1 - \frac{1}{4^k} \right)$$

apply to the k th inner ring.

Thus, for the innermost ring, which, by the way, reduces to one point for even n , the bounds

$$-M_1 \left(\frac{1}{4^{n_1}} \right) \text{ and } M_2 \left(1 - \frac{1}{4^{n_1}} \right), \quad n_1 = \frac{n}{2} \text{ resp. } \frac{n-1}{2}$$

are obtained after performing the first improvement from the outside to the inside.

In any case, after the first fully implemented improvement, all values lie between the bounds

$$-M_1\alpha \text{ and } M_2\alpha,$$

where α is a true fraction. The next improvement reduces the bounds to

$$-M_1\alpha^2 \text{ and } M_2\alpha^2,$$

and thus the convergence becomes clear, which of course progresses much more rapidly than this extremely crude estimation indicates.

But how to choose the first values? The example §2, 3 gives the answer to this question. In the case $n = 4$, where due to the weight table (§2, 1) the calculation can easily be done directly, first set

$$u_{2,2}^0 = \frac{1}{4} (a_{0,2} + a_{2,0} + a_{4,2} + a_{2,4}),$$

then calculate

$$u_{1,1}^0 = \frac{1}{4} (a_{0,0} + a_{0,2} + a_{2,0} + u_{2,2}^0),$$

etc. on the diagonal intersections of the four partial squares, then

$$u_{1,2}^0 = \frac{1}{4} (a_{0,2} + u_{1,1}^0 + u_{1,3}^0 + u_{2,2}^0),$$

etc. and finally average, in each case using the nearest neighboring values. However, it is possible to go a few steps further with the preliminary interpolation without any damage, i.e. to progress from $n = 4$ to $n = 8, 16, \dots$ without moving far away from the real sought values. The example discussed above §2, 3 was treated which, calculated to four decimals by this method, showed agreement to within two decimal units when calculated five times. Of course, symmetry properties of the function greatly facilitate the calculation. If, for example, on a square a boundary value $+1$ is prescribed for one pair, -1 for the other pair on opposite sides, then on the diagonal the sought function is zero and it is sufficient to calculate for an octant, for which in a very short time, you will find the table:

0.789	0.784	0.766	0.733	0.677	0.579	0.395	0
0.591	0.581	0.549	0.490	0.395	0.243	0	
0.414	0.401	0.359	0.285	0.170	0		
0.266	0.251	0.205	0.122	0			
0.150	0.133	0.084	0				
0.066	0.050	0					
0.016	0						
0							

which is sufficient for $15^2 = 225$ interior points. The large change in the corner is explained by the discontinuity of the prescribed values jumping from $+1$ to -1 on the boundary. By the way, it is easy to see with the aid of the function theory that, if the angle which a ray starting from the corner in with the upper side of the square is denoted by φ , then

$$u = 1 - \frac{4\varphi}{\pi}$$

is to be set near the corner and thus the directions are known under which the lines $u = \text{const}$ start from the corners.

In such places one can also refine the mesh, i.e. by solving the boundary value problem again for a smaller area, with on the one hand the given boundary values, on the other hand, as far as the new boundary runs in the interior of the basic area, the values of the calculations carried out up to then, supplemented by graphic and numerical interpolation.

[Analytically, if $u(x, y)$ is a first approximation, a function given for every point, the procedure could be refined by placing around each point in the

interior a small circle which does not exceed the boundary, and $u(x, y)$ is replaced each time by the mean value of the function on the circle boundary. One would then have the radii of these circles converge to zero to obtain the harmonic function. In this way the basic idea of Boltzmann could be transformed to become usable for analysis.]

We have only talked about the square so far. If the boundary is then a staircase polygon, then basically nothing changes in the proof of convergence, only the number $1/4$ is replaced by the number $1/8$, from which it can be seen that also mathematically the convergence will progress somewhat more slowly.

If the boundary is a drawn or analytically given curve, the difference equation must be changed in an easy way. For example, if $P(x, y)$, $P_1(x + \varepsilon, y)$, $P_2(x, y - \varepsilon)$ are points in the interior with values u^0, u_1^0, u_2^0 and the parallels to the axes passing through P intersect the boundary curve at the points $S_1(x - a_1\varepsilon, y)$ and $S_2(x, y - a_2\varepsilon)$, respectively, where a_1 and a_2 are between zero and one, then furthermore, if the boundary values a_1 and a_2 are prescribed at these points, the improved value in P is

$$u' = \frac{1}{2} \left\{ \frac{a_1 + a_1 u_1^0}{1 + a_1} + \frac{a_2 + a_2 u_2^0}{1 + a_2} \right\}.$$

2. The determination of the conjugate function.

[...]

3. The Richardson Investigations. The completeness alone requires a statement on the extensive and meritorious work of Richardson which I have received only when my train of thought was established in all details. The same basic idea is found in Richardson, viz. the reduction of the differential equation to a difference equation, which was also used by Runge in his work cited several times, but then the establishment and considerations, which likewise pursue the goal "starting from a table of values, which exactly fulfills the boundary conditions and in the interior (for inner grid points) comes as close as possible, to modify the table according to certain methods and to cause that it approaches the integral of the difference equation indefinitely" (op. cit., p. 318).

His way, however, is a completely different one, not Jacobi's so simple and handy approximation method. His method extends to difference equations what results from the requirement "that a certain homogeneous quadratic positive function becomes a minimum".

In any case, our method based on the work of Jacobi is also valid for all differential equations arising from the requirement that

$$\delta \iint \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + k^2 u^2 \right) dx dy$$

or any other by coordinate transformation. For on the one hand coordinate transformation cannot change its validity, on the other hand it is valid, as we will see shortly, for the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = k^2 u,$$

following from the aforementioned variational problem.

Here one obtains the extended difference equation

$$u(x - \varepsilon, y) + u(x + \varepsilon, y) + u(x, y - \varepsilon) + u(x, y + \varepsilon) = (4 + \varepsilon^2 k^2)u(x, y)$$

and if one uses it in the sense of Jacobi for the correction of the values, one soon recognizes that the convergence is even faster than for the case $k^2 = 0$. - By the way, the field of application is certainly not yet exhausted with the cases listed here. -

Richardson deals mainly with physical-technical problems, e.g. the pressure conditions in dams, and substantiates his methods by extensive tables⁸.

Introduction of general isothermal meshes, calculation of pairs of conjugate harmonic functions, and finally the inclusion of conformal mapping, which is to be given special emphasis, was far from the English author's goals.

§4. The conformal mapping.

[...]

This is how Boltzmann's ideas turn out to be practical and, according to a remark by Jacobi about the approximate resolution of a system of linear equations, formed into a convergent procedure, as a source of more promising and, as we hope to have shown, already tried and tested methods of calculation, which on the one hand make a contribution to applied mathematics, on the other hand find support in individual cases, but also provide inspiration in newly opened, still lively development studies of the theory of functions.

⁸The following passage provides information about the resources of the English researcher, which would only be lost through any attempt at translation and which should therefore be quoted in the original language: "So far I have paid piece rates for the operation D_2u of about $n/18$ pence per coordinate point, n being the number of digits. The chief trouble to the computers has been the intermixture of plus and minus signs. As to the rate of working, one of the quickest boys [!!] averaged 2000 operations D_2u per week, for number of three digits, those done wrong being discounted".