# On Seidel's method to solve linear equations with a very large number of unknowns by successive approximation.

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Extract of a letter from Professor Mehmke to Professor Nekrasov.

As I have now seen, my investigations are in a different direction than yours. I take the opportunity to tell you a few things about it. It was important to me to do a service to the electrical engineers. When calculating electrical central systems, they often have systems of up to 30 linear equations to be solved. However, these systems are usually not normal. Mr. Seidel has demonstrated the convergence of his method only for normal systems and suggested to bring other systems systems to the normal form. This effort can be saved in most cases. It was clear from the beginning that Seidel's method must also be applicable to a general system provided that certain conditions are fulfilled. I have now found that when applying Seidel's method to the system

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n + c_1 &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n + c_2 &= 0 \\ \vdots &= \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n + c_n &= 0 \end{aligned}$$

convergence occurs, if for $i = 1, 2, \ldots$, the absolute value of $a_{i,i}$ is at least as large as the sum of the absolute values of $a_{1,i}, a_{2,i}, \ldots, a_{i-1,i}, a_{i+1,i}, \ldots, a_{n,i}$. Of course, this is a sufficient but not necessary condition for convergence, This condition, as experience has shown me, is almost always fulfilled by the systems of equations occurring in practice.

That Seidel's method often converges very slowly to the solution, I have seen in various practical examples. I have therefore looked for means to improve the convergence of Seidel's method. The following method has proved perfectly practical for me. I decompose the whole system into groups of 2 or more equations. Such a group consisting of only 2 equations like:

$$a_{k,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,n}x_n + c_k = 0$$

$$a_{\ell,1}x_1 + a_{\ell,2}x_2 + \cdots + a_{\ell,n}x_n + c_\ell = 0$$

By inserting any initial values for the unknowns into these equations, the left-hand sides may take the values $f_k$, reps. $f_\ell$. Then I determine the improvements $\Delta x_k, \Delta x_\ell$ of the unknowns $x_k$ and $x_\ell$ by solving the two equations

$$\begin{aligned} a_{k,k}\Delta x_k + a_{k,\ell}\Delta x_\ell + f_k &= 0, \\ a_{\ell,k}\Delta x_k + a_{\ell,\ell}\Delta x_\ell + f_\ell &= 0, \end{aligned}$$

then I go to the next group of unknowns and so on. For practical reasons, each group will consist of at most 3 equations, in general less. It is easy to see, that in this way one will approach the goal extraordinarily much faster than with Seidel's method in its original form. For the convergence, it is sufficient that the absolute value of the determinant

$$\begin{vmatrix} a_{k,k} & a_{k,\ell} \\ a_{\ell,k} & a_{\ell,\ell} \end{vmatrix}$$

is at least as large as the sum of the absolute values of all determinants of the form

$$\begin{vmatrix} a_{k,k} & a_{k,\ell} \\ a_{h,k} & a_{h,\ell} \end{vmatrix}$$

and at the same time at least as large as the sum of the absolute values of all determinants of the form

$$\begin{vmatrix} a_{h,k} & a_{h,\ell} \\ a_{\ell,k} & a_{\ell,\ell} \end{vmatrix}$$

where $h$ has to assume all values from 1 to $n$ - except for the values $k$ and $\ell$.

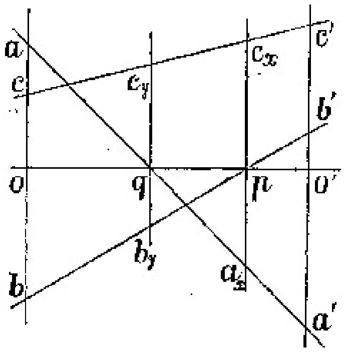
Since it would be useless to calculate the improvements of the unknowns with more than about 2 digits for each single step, so here a convenient graphical method exists. I use a method originating from Prof. van den Berg, whose application to a system of 2 equations of the form

$$\begin{aligned} ax + by + c &= 0, \\ a'x + b'y + c' &= 0 \end{aligned}$$

is as follows.

On two axes parallel to each other, take the zero points $O$ and $O'$, the units of longitude and the positive directions as desired, plot on the first axis from $O$ the coefficients $a, b, c$ as distances in the correct scale and way. Also on the second axis the coefficients $a', b', c'$ from $O'$, intersect $O\,O'$ with $b\,b'$ and with $a\,a'$ giving respectively $p$ and $q$. Draw parallels with the axes, intersections of the $p$ line with $a\,a'$ and $c\,c'$ give $a_x$ and $c_x$, intersections of the $q$ line with $b\,b'$ and $c\,c'$ give $b_y$ and $c_y$, then the solution is

$$x = -\frac{pc_x}{pa_x}, \quad y = -\frac{qc_y}{pb_y}.$$




When applied to our original task, one enjoys the great advantage that with the repeated correction of the same group of unknowns, the coefficients of the same group remain unchanged and therefore the major part of the lines need not be repeated. In the case just considered of a group of two equations, for example, since only $c$ and $c'$ change, for the second time only the drawing of a single straight line, namely $c\,c'$, is sufficient for the determination of the unknowns.

I still take the liberty to point out an incorrect statement in Günther's *Textbook of Determinants*, 2nd edition, page 105. The author says there at the end of §5: "The same (namely the procedure of Seidel) is only to be regarded as a convergent one with complete certainty, if each subdeterminant codiagonal to the determinant of the system is positive". One can be convinced by the following example of the incorrectness of this assertion. The given system is:

$$\begin{aligned} x - 0.9y - 0.8z - 2 &= 0 \\ -0.8x + y - 0.8z + 9 &= 0 \\ -0.6x - 0.5y + z + 9 &= 0 \end{aligned}$$

The solution is $x = 15$, $y = 10$, $z = 5$. Here, every subdeterminant whose diagonal forms a piece of the diagonal of the system is positive. Nevertheless, as the experiment shows, divergence is present.