

Determination of the unknowns by the method of least squares with a very large number of unknowns

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1.

Astronomy and geodesy represent cases when we have to determine by the method of least squares a large number of unknowns. In these cases the solution of the normal system of equations used to determine the unknowns present enormous difficulties. The normal system sometimes contains up to 70 unknowns. The determinants by means of which these unknowns are expressed, each containing 70 rows, will consist of an enormously large number of terms expressed by the product $1.2.3 \cdots 70$. Calculation of the value of such a determinant is in general unthinkable. In order to bypass these difficulties, astronomers resort to an approximate calculation of the unknowns. Of the methods of this kind, Ludwig Seidel's way is considered the most convenient, consisting in successive approximations to the solution.

The author's memoir has the title: *Ueber ein Verfahren die Gleichungen, auf welche die Methode der kleinsten Quadrate führt, sowie lineäre Gleichungen überhaupt, durch successive Annäherung aufzulösen* (Aus den Abhandlungen der k. bayer. Academic der W. II. Cl. XI Bd. III Abth. 1874. Munchen).

Reviewing Seidel's method at the suggestion of the respected astronomer V.K. Cerasky, who has to deal with these difficulties, I noticed that in his memoir Seidel does not touch a very important point in practical respect, that is, the question about the speed with which it is possible to approach the searched solutions. To remedy to this shortcoming, I will show that, under the right circumstances, Seidel's method quite quickly approach the solutions we are looking for, but very often there can be cases when this convergence will be slow, even infinitely slow.

§2 Basics of the Seidel Method

Let x, y, z, \dots, t be μ unknowns. Assuming

$$f_i = a_i x + b_i y + c_i z + \cdots + p_i t - q_i,$$

let us assume that the equations obtained on the basis of the observations are:

$$f_1 = 0, f_2 = 0, f_3 = 0, \dots, f_m = 0,$$

with $m \geq \mu$. The normal system of equations obtained by the method of least squares, as is well known, is as follows (1):

$$\begin{aligned} (aa)x + (ab)y + (ac)z + \dots + (ap)t - (aq) &= 0 \\ (ba)x + (bb)y + (bc)z + \dots + (bp)t - (bq) &= 0 \\ (ca)x + (cb)y + (cc)z + \dots + (cp)t - (cq) &= 0 \\ &\vdots = \vdots \\ (pa)x + (pb)y + (pc)z + \dots + (pp)t - (pq) &= 0 \end{aligned}$$

where, in fact,

$$(k\ell) = k_1\ell_1 + k_2\ell_2 + \dots + k_m\ell_m.$$

Values x, y, z, \dots, t satisfying equations (1), as it is known, give the minimum of

$$Q = \frac{1}{2}(f_1^2 + f_2^2 + \dots + f_m^2). \quad (2)$$

By decomposing the right-hand side of this equality by the powers of x , we find:

$$Q = \frac{1}{2} \left\{ (aa)x^2 + 2(ab)xy + 2(ac)xz + \dots + 2(ap)xt - 2(aq)x \right\} + R, \quad (2')$$

where R is a polynomial that does not contain x . Equality (2') can be written as:

$$Q = \frac{1}{2(aa)} \left\{ (aa)x + (ab)y + (ac)z + \dots + (ap)t - (aq) \right\}^2 + P, \quad (3)$$

where P is a polynomial that does not contain x .

Let $x_0, y_0, z_0, \dots, t_0$ be any system of particular variables x, y, z, \dots, t that do not satisfy equation (1). The value corresponding to these variables is:

$$Q = \frac{1}{2(aa)} \left\{ (aa)x_0 + (ab)y_0 + (ac)z_0 + \dots + (ap)t_0 - (aq) \right\}^2 + P_0. \quad (3')$$

Let x_1 be the value that satisfies the equation

$$(aa)x_1 + (ab)y_0 + (ac)z_0 + \dots + (ap)t_0 - (aq) = 0.$$

Obviously, for the system of particular values of $x_1, y_0, z_0, \dots, t_0$ of variables x, y, z, \dots, t the value of Q is equal to Q_1 and it follows:

$$Q_1 < Q_0.$$

Thus, for the values $x_1, y_0, z_0, \dots, t_0$, the value of Q is closer to its minimum than for $x_0, y_0, z_0, \dots, t_0$.

Let there be a value y_1 that satisfies the equation

$$(ba)x_1 + (bb)y_1 + (bc)z_0 + \dots + (bp)t_0 - (bq) = 0.$$

In this way, it is easy to see that the value of Q , which can be represented in the form:

$$Q = \frac{1}{2(bb)} \{(ba)x + (bb)y + (bc)z + \dots + (bp)t - (bq)\}^2 + L,$$

for the variables $x_1, y_1, z_0, \dots, t_0$ will be closer to the minimum than the value of Q corresponding to the values $x_1, y_0, z_0, \dots, t_0$. Continuing our reasoning in this way, we can successively replace all $x_0, y_0, z_0, \dots, t_0$ with $x_1, y_1, z_1, \dots, t_1$, constantly approximating the minimum of Q . Then, according to the same method, we can replace the system of values $x_1, y_1, z_1, \dots, t_1$ with the system of values $x_2, y_2, z_2, \dots, t_2$ etc. In the limit, after an uncountable set of such operations, the quantity Q will approach its minimum, and the system of the obtained values of the variables x, y, z, \dots, t will approach to values satisfying equations (1).

§3 Expressions of errors of approximate values obtained by Seidel's method.

Seidel's method in the limit undoubtedly approximates the required solutions of the system (1), but to judge the speed of this approximation it is necessary to have general expressions of approximation errors, obtained by this method. These expressions that we will deduce, are missing from Seidel's work.

In deducing these expressions, we will assume that the iterative approximations are computed in the following natural order, one after the other. First of all, the quantities x_1, y_1, \dots, t_1 are found one after another and then, x_2, y_2, \dots, t_2 in the same order, etc¹

Let $x_n, y_n, z_n, \dots, t_n$ be a system of approximate values, obtained by Seidel's method from the above mentioned operations. The system of values obtained by the same method after finding $x_n, y_n, z_n, \dots, t_n$ will be $x_{n+1}, y_{n+1}, z_{n+1}, \dots, t_{n+1}$. These quantities satisfy the equations (4):

$$\begin{aligned} (aa)x_{n+1} + (ab)y_n + (ac)z_n + \dots + (ap)t_n - (aq) &= 0 \\ (ba)x_{n+1} + (bb)y_{n+1} + (bc)z_n + \dots + (bp)t_n - (bq) &= 0 \\ (ca)x_{n+1} + (cb)y_{n+1} + (cc)z_{n+1} + \dots + (cp)t_n - (cq) &= 0 \\ &\vdots = \vdots \\ (pa)x_{n+1} + (pb)y_{n+1} + (pc)z_{n+1} + \dots + (pp)t_{n+1} - (pq) &= 0 \end{aligned}$$

¹This order is the usual and simplest, although it can be deviated from. These deviations, which speed up the approximation a bit, will be discussed at the end of the article.

These equations can be viewed as difference equations with finite differences with respect to the variable n . One of the functions $x_n, y_n, z_n, \dots, t_n$ of n is easily excluded by the last of the equations (4), and by elimination we will have a system of $\mu-1$ equations with $\mu-1$ first-order functions with constant coefficients. The solution of these equations, which does not theoretically present difficulties, leads to the required expression of errors for the values $x_n, y_n, z_n, \dots, t_n$.

Let x', y', z', \dots, t' be the exact values satisfying equations (1). Let the errors corresponding to the approximate values of x_n, y_n, \dots be:

$$\xi_n = x_n - x', \quad \eta_n = y_n - y', \quad \zeta_n = z_n - z', \dots, \quad \tau_n = t_n - t'.$$

These quantities satisfy the equations (4')

$$\begin{aligned} (aa)\xi_{n+1} + (ab)\eta_n + (ac)\zeta_n + \dots + (ap)\tau_n &= 0 \\ (ba)\xi_{n+1} + (bb)\eta_{n+1} + (bc)\zeta_n + \dots + (bp)\tau_n &= 0 \\ (ca)\xi_{n+1} + (cb)\eta_{n+1} + (cc)\zeta_{n+1} + \dots + (cp)\tau_n &= 0 \\ &\vdots = \vdots \\ (pa)\xi_{n+1} + (pb)\eta_{n+1} + (pc)\zeta_{n+1} + \dots + (pp)\tau_{n+1} &= 0 \end{aligned}$$

Let us look for a particular solution of this system of equations, assuming

$$\xi_n = A\alpha^n, \quad \eta_n = AB\alpha^n, \quad \zeta_n = AC\alpha^n, \quad \dots, \quad \tau_n = AP\alpha^n. \quad (5)$$

Substituting these quantities into equation (4'), after simplification, we obtain (6):

$$\begin{aligned} (aa)\alpha + (ab)B + (ac)C + \dots + (ap)P &= 0 \\ (ba)\alpha + (bb)\alpha B + (bc)C + \dots + (bp)P &= 0 \\ (ca)\alpha + (cb)\alpha B + (cc)\alpha C + \dots + (cp)P &= 0 \\ &\vdots = \vdots \\ (pa) + (pb)\eta_{n+1}B + (pc)C + \dots + (pp)P &= 0 \end{aligned}$$

Excluding from this system the quantities B, C, \dots, P , we obtain the following equation (7) for determining α : [the determinant must be zero to get a solution]

$$\begin{vmatrix} (aa)\alpha & (ab) & (ac) & \dots & (ap) \\ (ba)\alpha & (bb)\alpha & (bc) & \dots & (bp) \\ (ca)\alpha & (cb)\alpha & (cc)\alpha & \dots & (cp) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (pa) & (pb) & (pc) & \dots & (pp) \end{vmatrix} = 0.$$

This equation of degree $\mu-1$ with respect to α admits $\mu-1$ roots $\alpha_1, \alpha_2, \dots, \alpha_{\mu-1}$. Each of them will correspond to a specific system of values B, C, \dots, P , determined by equations (6). There are $\mu-1$ partial solutions of the form (5)

with an arbitrary constant A . These partial solutions make it easy to figure out the general solution (8) that will be:

$$\begin{aligned}
\xi_n &= A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_{\mu-1}\alpha_{\mu-1}^n \\
\eta_n &= A_1B_1\alpha_1^n + A_2B_2\alpha_2^n + \cdots + A_{\mu-1}B_{\mu-1}\alpha_{\mu-1}^n \\
\zeta_n &= A_1C_1\alpha_1^n + A_2C_2\alpha_2^n + \cdots + A_{\mu-1}C_{\mu-1}\alpha_{\mu-1}^n \\
&\vdots = \vdots \\
\tau_n &= A_1P_1\alpha_1^n + A_2P_2\alpha_2^n + \cdots + A_{\mu-1}P_{\mu-1}\alpha_{\mu-1}^n
\end{aligned}$$

where $A_1, A_2, \dots, A_{\mu-1}$ are arbitrary constants, $\alpha_i, B_i, C_i, \dots, P_i$ are the quantities determined by equations (6) and α satisfies equation (7). The arbitrary constants $A_1, A_2, \dots, A_{\mu-1}$ are related to the initial errors $\eta_0, \zeta_0, \dots, \tau_0$ and with the initial arbitrarily chosen values y_0, z_0, \dots, t_0 by means of equations (9):

$$\begin{aligned}
\eta_0 &= y_0 - y' = A_1B_1 + A_2B_2 + \cdots + A_{\mu-1}B_{\mu-1} \\
\zeta_0 &= z_0 - z' = A_1C_1 + A_2C_2 + \cdots + A_{\mu-1}C_{\mu-1} \\
&\vdots = \vdots \\
\tau_0 &= t_0 - t' = A_1P_1 + A_2P_2 + \cdots + A_{\mu-1}P_{\mu-1}
\end{aligned}$$

If the equation (7) has equal roots, then the right-hand sides of equations (8) must be changed in a certain way, but the conclusions above still hold.

§4 The speed with which Seidel's method brings us closer to the solutions we are looking for.

Since in the limit at $n \rightarrow \infty$ the errors of the quantities x_n, y_n, z_n, \dots should, according to the proof in §2, tend to zero, all roots $\alpha_1, \alpha_2, \dots, \alpha_{\mu-1}$ of equation (7) must have a modulus less than unity. Only under these conditions will the terms of the right-hand sides of the equations (8) tend to zero as n increases.

Let α be the root of equation (7) of largest modulus, which we will call the main root. The error terms corresponding to the root α of $\xi_n, \eta_n, \zeta_n, \dots$, defined by equations (8), will be:

$$A\alpha^n, AB\alpha^n, AC\alpha^n, \dots$$

The value of A , which depends, as equations (9) show, on $y_0 - y', z_0 - z', \dots, t_0 - t'$, cannot be set in advance, since y', z', \dots, t' are assumed to be unknown.

Therefore, if we choose y_0, z_0, \dots, t_0 arbitrarily, A can only accidentally be zero, and, in general, its value will be nonzero, and the error terms $A\alpha^n, AB\alpha^n, AC\alpha^n, \dots$ will decrease with the greatest slowness in comparison with the other terms of the errors. The errors of the approximated values

obtained by Seidel's method will quickly decrease if the modulus of the main root α is small. If the modulus of the main root happens to be close to 1, which is not uncommon, and if the unlikely circumstance corresponding to the choice of such initial values y_0, z_0, \dots, t_0 giving $A = 0$ does not happen, then Seidel's method will slowly approach the solutions of equation (1). In general, the speed with which Seidel's method approximates the solutions we are looking for is quite conditional on the size of the roots of the equation (7). If all the roots of the equation (7) without excluding the main one are small, this method will quickly approach the solutions we are searching for. Otherwise the approximation will be slow.

Let us give examples of both kinds.

Example 1.

$$\begin{aligned}x - 3y + z - 3 &= 0 \\2x + y - 4z &= 0 \\x + 1.01y + 7.1z - 9.1 &= 0\end{aligned}$$

The normal system (1) with these data is:

$$\begin{aligned}6x + 0.01y + 0.1z - 1.21 &= 0 \\0.01x + 11.0201y + 0.171z - 0.191 &= 0 \\0.1x + 0.171y + 67.41z - 67.61 &= 0\end{aligned}$$

Equation (7) looks like this:

$$4457.189646 \alpha^2 - 0.28657801 \alpha + 0,000171 = 0$$

and has two imaginary conjugate roots. The modulus of each of them is $\rho = 0.0000196$. Seidel's method will approximate the solutions very quickly. Assuming, for example, $y_0 = 10, z_0 = 3$, we find:

$$x_1 = 1.95, \quad y_1 = -0.0309888, \quad z_1 = 1.000152.$$

These values are already quite close to the exact solution which is $x = 2, y = 0$ and $z = 1$. Applying Seidel's method a second time, we obtain the following very precise values:

$$x_2 = 2.0000491, \quad y_2 = 0.00000240, \quad z_2 = 0.99999993.$$

Example 2.

$$\begin{aligned}x + y + z - 6 &= 0 \\2x + y + z - 7 &= 0 \\x + 2y + z - 8 &= 0\end{aligned}$$

The system of normal equations is

$$\begin{aligned} 6x + 5y + 4z - 28 &= 0 \\ 5x + 6y + 4z - 29 &= 0 \\ 4x + 4y + 3z - 21 &= 0 \end{aligned}$$

Equation (7) looks like this:

$$108\alpha^2 - 187\alpha + 80 = 0.$$

The roots of this equation are:

$$\alpha_1 = 0.959\dots, \quad \alpha_2 = 0.872\dots,$$

The main root is very close to 1. Therefore the convergence of Seidel's method will be very slow. Assuming, for example, $y_0 = 3$, $z_0 = 4$, we find

$$\begin{aligned} x_1 &= -0.5, & y_1 &= 2.58333\dots, & z_1 &= 4.22222\dots, \\ x_2 &= -0.300925\dots, & y_2 &= 2.26929\dots, & z_2 &= 4.37557\dots, \\ x_3 &= -0.141455\dots, & y_2 &= 2.04095\dots, & z_2 &= 4.47638\dots, \end{aligned}$$

These values differ significantly from the exact solution, which is: $x = 1$, $y = 2$, $z = 3$.

The system of equations (10)

$$f_1 = 0, f_2 = 0, \dots, f_m = 0$$

can be chosen so that the modulus of the main root of equation (7) will be as close to one as we wish. In fact, let there exist between the functions f_1, f_2, \dots, f_m , $m - \mu + p$ linear relations of the form:

$$\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_m f_m = 0,$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are constant quantities. If $\mu \geq 1$, then the system of equations (10) will be undefined. At the same time the normal system (1) will be indeterminate, and we will have (11):

$$\begin{vmatrix} (aa) & (ab) & (ac) & \dots & (ap) \\ (ba) & (bb) & (bc) & \dots & (bp) \\ (ca) & (cb) & (cc) & \dots & (cp) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (pa) & (pb) & (pc) & \dots & (pp) \end{vmatrix} = 0$$

Consequently, equation (7) will have a root equal to 1. Give to the values of $a_1, a_2, \dots, b_1, b_2, \dots, c_1, c_2, \dots, p_1, p_2, \dots$ very small increments, so that the

normal system (1) is transformed to a definite one. At the same time the determinant in the left-hand side of equality (11), will be a nonzero value, though very small, and the equation (7) will have a root very close to one. Thus, we obtain an artificial system for which the main root of the equation (7) will be as close to one as we wish. Such artificial systems, with the main root of the equation (7) as close to one as we wish, can be constructed in many number of ways, for example so that the principal root is close to -1 or to some imaginary quantity whose modulus is 1. Consequently, systems of this kind can be encountered very often. Undoubtedly, cases of this kind can occur for systems derived from observations. Seidel's method is not suitable in such cases.

Let us also note that equation (7) can be represented as (7'):

$$(aa)(bb)(cc)\cdots(pp)\alpha^{\mu-1} + \cdots + (ab)(bc)(cd)\cdots(pa) = 0.$$

So, it is true that,

$$\alpha_1\alpha_2\cdots\alpha_{\mu-1} = \pm \frac{(ab)(bc)\cdots(pa)}{(aa)(bb)\cdots(pp)}.$$

This equality shows that

$$\text{mod } \alpha > K \quad (12)$$

where α is the main root of equation (7) and

$$K = \text{mod } \sqrt[\mu-1]{\frac{(ab)(bc)\cdots(pa)}{(aa)(bb)\cdots(pp)}}. \quad (13)$$

If the value of K turns out to be insufficiently small, then Seidel's method will, obviously, slowly approximate the solution we are looking for. Finally, let us note that Seidel's method quickly approaches solutions when the values $(aa), (bb), (cc), \dots, (pp)$ are quite large in comparison with each of the other coefficients in equations (1). In fact, in this case, the coefficient of the first term of equation (7') will be very large compared to the coefficients of the other terms of the same equation, and consequently all roots of equation (7') or (7) will be very small.

The order of finding the approximated values by Seidel's method can be taken differently from the one specified in §3. The approximation to the quantities we are looking for may seem to be somewhat accelerated. But when the main root of the equation (7) is very close to one, this technique will not make Seidel's method fast enough to be used in practice. To prove this point, I will look at an example in which, with the most advantageous order of calculation, Seidel's method approximates the solutions we are looking for very slowly.

§5 The speed of convergence of Seidel's method for the most advantageous order of calculations is unknown.

With respect to the speed of convergence of Seidel's method, the most advantageous order of calculation is the one in which the quantity Q defined by equality (2) decreases in the fastest way. Let us consider this order of calculations. Let the left-hand sides of equation (1) when replacing in them the values of x, y, \dots by x_0, y_0, \dots , be

$$\begin{aligned} (aa)x_0 + (ab)y_0 + \dots + (ap)t_0 - (aq) &= N_1 \\ (ba)x_0 + (bb)y_0 + \dots + (bp)t_0 - (bq) &= N_2 \\ &\vdots = \vdots \end{aligned}$$

If the first correction according to Seidel's method refers to the value of x_0 , then this correction will be:

$$x_1 - x_0 = -\frac{N_1}{(aa)},$$

and, as equality (3') shows, the value of Q will decrease by

$$M_1 = \frac{N_1^2}{2(aa)}.$$

If the first correction will relate to the value y_0 then the value of Q will be decreased by the value

$$M_2 = \frac{N_2^2}{2(bb)},$$

And so on. After calculating the values $M_1, M_2, M_3, \dots, M_\mu$ choose the maximal value. Let us assume that this maximal value will be

$$M_3 = \frac{N_3^2}{2(cc)}.$$

Obviously, the first correction is the most advantageous to refer to the value of z_0 . Having made this correction, we will have:

$$z_1 = z_0 - \frac{N_3}{(cc)}.$$

Let the left-hand sides of equation (1) with x, y, z, \dots, t replaced by the values $x_0, y_0, z_1, \dots, t_0$ be $N'_1, N'_2, \dots, N'_\mu$. Let

$$M'_1 = \frac{[N'_1]^2}{2(aa)}, \quad M'_2 = \frac{[N'_2]^2}{2(aa)}, \quad \dots \quad M'_\mu = \frac{[N'_\mu]^2}{2(aa)}.$$

The second correction of $x_0, y_0, z_1, \dots, t_0$ should be related to the largest of the values

$$M'_1, M'_2, \dots, M'_\mu,$$

and so on².

When calculating in this way, Seidel's method will somewhat approach faster to the desired solutions, but the but the gain in speed will not be great. In fact, if the modulus of the main root of equation (7) is rather close to 1, then after a few Seidel iterations the quantities M_1, M_2, \dots, M_μ are too insignificant and the largest of them only slightly decreases the value of Q , despite the considerable errors in the values being sought. Let us illustrate this with a numerical example.

$$\begin{aligned} 10x + y + z - 15 &= 0 \\ 20x + 2y + z - 27 &= 0 \\ 31x + 3y + 2z - 43 &= 0 \end{aligned}$$

The normal system for these data is as follows (14):

$$\begin{aligned} 1461x + 143y + 92z - 2023 &= 0 \\ 143x + 14y + 9z - 198 &= 0 \\ 92x + 9y + 6z - 128 &= 0 \end{aligned}$$

With these data equation (7) is:

$$122724\alpha^2 - 241127\alpha + 118404 = 0.$$

This equation has a root that is very close to 1. The solution of equation (14) is:

$$x = 1, \quad y = 2, \quad z = 3.$$

Let $x_0 = 0.9$, $y_0 = 1.9$, $z_0 = 2.9$. We find

$$N_1 = -169.6, \quad N_2 = -16.6, \quad N_3 = -10.7,$$

$$M_1 = 9.843997, \quad M_2 = 9.8411, \quad M_3 = 9.5408.$$

The first correction should refer to the value x_0 , and we will have:

$$x_1 = x_0 + 0.1160849 = 1.0160849.$$

Next we find:

$$N'_1 = 0.0000389, \quad N'_2 = 0.0001407, \quad N'_3 = -0.0201892,$$

$$M'_1 = 0, \quad M'_2 = 0.0000000007\dots, \quad M'_3 = 0.000034\dots$$

²In practical terms, such an order of calculations is not advantageous since it complicates calculations, because in addition to the corrections we have to calculate a whole range of quantities denoted by the letter M .

The second correction should be related to z_0 and the value of Q will drop very insignificantly. By correcting we will have $z_1 = 2.9033649$ and we find:

$$N_1'' = 0.3096097, \quad N_2'' = 0.0304248, \quad N_3'' = 0.0000002,$$

$$M_1'' = 0.00003281, \quad M_2'' = 0.00003306, \quad M_3'' = 0.$$

The third correction should be related to the value y_0 and, again, the value of Q will drop insignificantly. The correction is

$$y_1 - y_0 = -0.0021732.$$

This correction will even move the approximate y value away from its exact value. Further we find:

$$N_1''' = -0.0011579, \quad N_2''' = 0, \quad N_3''' = -0.0195586,$$

$$M_1''' = \frac{46}{10^{11}}, \quad M_2''' = 0, \quad M_3''' = 0.0000318.$$

Then,

$$z_2 - z_1 = 0.0032598,$$

and so on. The insignificance of the corrections compared to the magnitude of the total errors shows that it is necessary to carry out long calculations to obtain approximate values that are only accurate to 0.01.

The minimum value of Q in the example at hand is zero. The value corresponding to x_0, y_0, z_0 is $Q_0 = 9.5845$. The first correction significantly reduced this value³. After the first correction we will have:

$$Q_1 = Q_0 - M_1 = 0.001003.$$

The following corrections cause insignificant changes of this value. In fact, the values of Q after the second, third and fourth corrections will be:

$$Q_2 = Q_1 - M_3' = 0.000969,$$

$$Q_3 = Q_2 - M_2'' = 0.000936,$$

$$Q_4 = Q_3 - M_3''' = 0.000904.$$

These values are almost equal to each other and slowly tend to zero.

³This is conditioned by the randomness of the choice of the initial values, x_0, y_0 and z_0 . It would be possible to choose these values so that the decrease of Q was insignificant from the very beginning. When Seidel's method is applied further, the rate of approximation becomes slower, because of the error terms corresponding to the main root. The choice of initial values has a great influence on the course of calculation. Sometimes it is possible to choose initial values artificially in such a way, that after a finite number of operations by the Seidel method the exact solution is found. But if the initial values are chosen at random, we cannot count on such unlikely results.