On the compensation of observations according to the method of least squares, Notes I and II

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Note I

The formulas contained in this Note, may in some cases (such as it will be particularly indicated in §4) be useful for the calculations necessary to the compensation of conditional observations. We will use these formulas for some theoretical discussions in another Note on this same subject.

We will indicate for simplicity with the notation

$$\begin{bmatrix} a & b & c & \cdots \\ x & y & s & \cdots \\ A & B & C & \cdots \end{bmatrix} = 0$$

the system of σ normal equations having as unknowns the σ quantities $x, y, s \ldots$, as known terms $A, B, C, \ldots$, and as coefficients the summations

$$\begin{matrix} [aa] & [ab] & [ac] & \cdots \\ [ab] & [bb] & [bc] & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

formed, according to the notation of Gauss, by means of the system, with σ columns and n rows

$$\begin{matrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & \cdots \end{matrix}$$

We will give to the symbols

$$\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \cdots \\ \alpha_2 & \beta_2 & \gamma_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \beta_n & \gamma_n & \cdots \end{matrix}$$

the same meanings attributed to them by Gauss in his *Theoria combinationis observationum etc.* Among the $\alpha, \beta, \gamma, \ldots$ and $a, b, c, \ldots$ we have the relations

$$(1) \quad \begin{aligned} \alpha_r &= a_r[\alpha\alpha] + b_r[\alpha\beta] + c_r[\alpha\gamma] + \cdots \\ \beta_r &= a_r[\alpha\beta] + b_r[\beta\beta] + c_r[\beta\gamma] + \cdots \\ \gamma_r &= a_r[\alpha\gamma] + b_r[\beta\gamma] + c_r[\gamma\gamma] + \cdots \\ &\cdots \end{aligned}$$

$$(2) \quad \begin{matrix} [a\alpha] = 1 & [a\beta] = 0 & [a\gamma] = 0 & \cdots \\ [b\alpha] = 0 & [b\beta] = 1 & [b\gamma] = 0 & \cdots \\ [c\alpha] = 0 & [c\beta] = 0 & [c\gamma] = 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

Let

$$(I) \quad \begin{aligned} a_1v_1 + a_2v_2 + \cdots + a_nv_n + A = 0 \\ b_1v_1 + b_2v_2 + \cdots + b_nv_n + B = 0 \\ c_1v_1 + c_2v_2 + \cdots + c_nv_n + C = 0 \\ \cdots \end{aligned}$$

$$(II) \quad \begin{aligned} d_1v_1 + d_2v_2 + \cdots + d_nv_n + D = 0 \\ e_1v_1 + e_2v_2 + \cdots + e_nv_n + E = 0 \end{aligned}$$

now be the condition equations (in number of σ +2) to which are related the unknown corrections $v_1, v_2, \ldots, v_n$ to be applied to a certain system of direct observations. The most likely values of these corrections are given by relations of the form

$$\lambda_r = a_rk_a + b_rk_b + c_rk_c + \cdots + d_rk_d + e_rk_e,$$

where the k's must be calculated by solving the system of $\sigma + 2$ normal equations

$$(3) \quad \begin{bmatrix} a & b & c & \cdots & d & e \\ k_a & k_b & k_c & \cdots & k_d & k_e \\ A & B & C & \cdots & D & E \end{bmatrix} = 0$$

We separate the proposed condition equations into two groups, to the first of which we will ascribe the equations (I), to the second the equations (II)[1], and form the normal equations relating only to the conditions of the system (I), i.e. the equations

$$(4) \quad \begin{bmatrix} a & b & c & \cdots \\ h_a & h_b & h_c & \cdots \\ A & B & C & \cdots \end{bmatrix} = 0$$

and, based on this system of normal equations, we hold for the letters $\alpha, \beta, \gamma, \ldots$ the meanings established in the previous paragraph, so that we have for example:

$$(5) \quad \begin{aligned} h_a &= -A[\alpha\alpha] - B[\alpha\beta] - C[\alpha\gamma] - \cdots \\ h_b &= -A[\alpha\beta] - B[\beta\beta] - C[\beta\gamma] - \cdots \end{aligned}$$

[1] By way of simplicity, we have assumed the system (II) to consist of only two equations, but this in no way limits the deductions that follow.

and so on.

We define

$$(6)\quad \begin{aligned} d_r &= \alpha_r[ad] + \beta_r[bd] + \gamma_r[cd] + \cdots + p_r \\ e_r &= \alpha_r[ae] + \beta_r[be] + \gamma_r[ce] + \cdots + q_r \end{aligned}$$

Multiplying the first of these by a_r, and adding it to the analogous equations which are obtained by varying r from 1 to n, we obtain, taking into account (2),

$$[ap] = 0.$$

Similarly

$$[bp] = 0, \quad [cp] = 0,$$

and so on,

$$(7)\quad [aq] = 0, \quad [bq] = 0, \quad [cq] = 0,$$

and so on. Similarly

$$(8)\quad \begin{gathered} [\alpha p] = [ap][\alpha\alpha] + [bp][\alpha\beta] + [cp][\alpha\gamma] + \cdots = 0, \\ [\beta p] = 0, \quad [\gamma p] = 0, \ldots \\ [\alpha q] = 0, \quad [\beta q] = 0, \quad [\gamma q] = 0, \ldots \end{gathered}$$

Now multiplying (6) by p, and performing the summation $[dp]$, we have, observing (8),

$$[dp] = [pp].$$

Similarly

$$(9)\quad [qd] = [qp], \quad [pe] = [pq], \quad [qe] = [qq].$$

Finally, multiplying the first of (6) by d, and performing the summation $[dd]$, then similarly calculating $[de]$, $[ee]$, taking into account (9), we have:

$$(10)\quad \begin{aligned} [dd] &= [\alpha d][ad] + [\beta d][bd] + \cdots + [pp] \\ [de] &= [\alpha e][ad] + [\beta e][bd] + \cdots + [pq] \\ [ed] &= [\alpha d][ae] + [\beta d][be] + \cdots + [qp] \\ [ee] &= [\alpha e][ae] + [\beta e][be] + \cdots + [qq] \end{aligned}$$

We observe again that, taking into account the relations (1), the quantities $[\alpha d]$ $[\beta d]$ e.g., which appear in the preceding formulae can be expressed as follows

$$(11)\quad \begin{gathered} [\alpha d] = [ad][\alpha\alpha] + [bd][\alpha\beta] + [cd][\alpha\gamma] + \cdots \\ [\beta d] = [ad][\alpha\beta] + [bd][\beta\beta] + [cd][\beta\gamma] + \cdots \\ \cdots \\ [\alpha e] = [ae][\alpha\alpha] + [be][\alpha\beta] + [ce][\alpha\gamma] + \cdots \\ \cdots \end{gathered}$$

Let us now consider the first σ normal equations of the system (8) and let us multiply the first of them by $[\alpha\alpha]$, the second by $[\alpha\beta]$, the third by $[\alpha\gamma]$

and so on. Adding up and remembering (5), we will have

$$
(12) \quad \begin{aligned}
k_a &= h_a - [\alpha d]k_d - [\alpha e]k_e, \\
k_b &= h_b - [\beta d]k_d - [\beta e]k_e, \\
k_c &= h_c - [\gamma d]k_d - [\gamma e]k_e, \\
&\cdots
\end{aligned}
$$

If then these expressions of $k_a, k_b, \ldots$ are substituted in the last two normal equations of the system (3), we see very quickly that the coefficients of k_d, k_e in these transformed equations become exactly equal, in virtue of (10), to $[pp]$, $[pg]$, $[pq]$, $[qq]$ respectively.

We finally define

$$
(13) \quad \begin{aligned}
[ad]h_a + [bd]h_b + [cd]h_c + \cdots + D = D' \\
[ae]h_a + [be]h_b + [ce]h_c + \cdots + E = E'
\end{aligned}
$$

and the last two normal equations, of the system (3), transformed in the way now said, will become therefore

$$
(14) \quad \begin{aligned}
[pp]k_d + [pq]k_e + D' = 0 \\
[pq]k_d + [qq]k_e + E' = 0
\end{aligned}
$$

The formulas developed here demonstrate how the resolution of the normal system (3) can be deduced from that of the normal system (4). In fact, once the system (4) is solved, the formulas (13), the normal equations (14) and the relations (12) give us without difficulty the values of the unknowns k of the system of normal equations (3).

The values of the coefficients $[pp]$, $[pq]$, $[qq]$ of (14) must be deduced directly from (10). As for the quantities $[\alpha d]$, $[\beta d]$,... that appear in (10), their values could certainly be derived from (11) when the summations

$$[\alpha\alpha],\ [\alpha\beta],\ [\alpha\gamma], \ldots, [\beta\beta], \ldots$$

are known. If the values of these summations are not known, the $[\alpha d]$, $[\beta d]$,..., $[\alpha e]$, $[\beta e]$, ..., can be obtained by solving the two systems of normal equations

$$
(15) \quad \begin{aligned}
[aa][\alpha d] + [ab][\beta d] + \cdots = [ad] \\
[ab][\alpha d] + [bb][\beta d] + \cdots = [bd] \\
\cdots
\end{aligned}
$$

$$
(15\,bis) \quad \begin{aligned}
[aa][\alpha e] + [ab][\beta e] + \cdots = [ae] \\
[ab][\alpha e] + [bb][\beta e] + \cdots = [be] \\
\cdots
\end{aligned}
$$

3. Concerning the normal system (14), it is good to observe how it can never lead to undetermined values of the unknowns, if, as we must assume,

the proposed condition equations are fully independent of each other. In fact, as is well known, the determinant of the equations (14)

$$\begin{vmatrix} [pp] & [pq] \\ [pq] & [qq] \end{vmatrix}$$

cannot be zero unless we have

$$p_1 = Hq_1, \quad p_2 = Hq_2, \ldots, p_n = Hq_n$$

where H is a constant.

On the other hand, equations (6), taking into account (1) and (11) can be written

$$\begin{aligned} d_r &= a_r[\alpha d] + b_r[\beta d] + c_r[\gamma d] + \cdots + p_r \\ e_r &= a_r[\alpha e] + b_r[\beta e] + c_r[\gamma e] + \cdots + q_r \end{aligned}$$

From these by virtue of (15) we can immediately deduce

$$\begin{aligned} d_r = He_r + \{[\alpha d] - H[\alpha e]\}a_r + \{[\beta d] - H[\beta e]\}b_r + \\ + \{[\gamma d] - H[\gamma e]\}c_r + \cdots \end{aligned} \tag{16}$$

If the determinant is zero, there must be a linear relation (16) between the coefficients of the various condition equations: that is, it is necessary that these equations are not all independent of each other, which is against our assumption.

4. As we noted earlier, the formulas of §2 can provide an indirect resolution of the normal system (4). The application of this indirect procedure presents the maximum convenience when, once compensated with the usual direct calculation a certain system of observations linked by a large number of equations of condition, it is necessary to redo the calculation for the addition of a few equations of condition, at first not considered. This can happen, of course, when some new observations are added to those previously made. In this case, applying our formulae, the partial compensation already accomplished is completely used, and the computational work to be added to complete the compensation is quite small, if, of course, the new conditions added are very small in number.

5. We will finally observe that the normal equations (14) are not new in the theory that occupies us. They are nothing else than one of Gauss' systems of reduced equations, and precisely the reduced system which is obtained by eliminating the first σ unknowns from the system (4).

We believe, however, that it is not without interest to have highlighted the principal relations which link the coefficients of these reduced equations; relations notable both for their practical usefulness in the case mentioned in the preceding §, and for the use we shall make of them in a following Note.

Note II

The compensation of geodetic observations, according to the method of least squares, is usually hampered by the excessive complication of calculations, which occurs whenever the number of condition equations is large. The prolixity of the calculations refers especially to the solution of normal equations, for which the work of formation and numerical solution increases approximately by the square of the number of conditions.

In order to simplify the calculations, we consider the artifice of dividing the condition equations into several categories applying to each one separately and subsequently the compensation calculation. In this way the solution of a certain group of normal equations (in number of σ) is substituted by the solution of many groups of equations (in number of $\alpha, \beta, \gamma, \ldots$ respectively, where $\alpha + \beta + \gamma + \ldots = \sigma$) and the overall work is considerably reduced. The procedure to be used is as follows.

Compensate the observations by taking into account only the first system (α) of conditions, consider the compensated values as given directly by the observations, and from the observations, and over them operate a new compensation according to (β) . Then by means of the system conditions (γ) operate a third compensation on the values already corrected by the previous two operations. And so proceed in such a way as to employ, one after the other, all the partial systems of conditions. After this we will say that we have completed a complete round of compensations. When the first round is completed, the values obtained will not generally satisfy the conditions of the system (α). In this case we will repeat the next compensation by means of the various systems of conditions by performing a second run, and so on until a compensated value system has been obtained, which satisfy all the conditions proposed.

In the 2nd, 3rd, and 4th rounds of compensation, the systems of normal equations to be solved do not differ from the corresponding systems of the first round except for the known terms, so that, for a practical calculator, the calculation of the subsequent rounds, after the first, is very simple and fast.

In order for this procedure to be rationally employed, it is necessary to demonstrate in general:

a) that the operation has a limit, that is, that it really tends to provide a system of corrections that simultaneously satisfy all the given condition equations,

b) that these definitive corrections coincide with those which would be provided by the direct calculation of compensation applied, in the usual way, to the whole set of proposed equations.

The proof of this second theorem was hinted at by Gauss[2] and then

[2] Supplementum theoriae combinationis observationum etc. pp. 18-20,

clearly developed by Mr. Helmert[3]; it does not present any difficulty. In the present Note we propose mainly to prove the first assertion, which is not as obvious as the second one.

We will begin, for this purpose, by deducing the fairly simple system of formulas, for which the above calculation of successive approximations can practically be made.

3. We will consider only two sets of condition equations, and, for the sake of saving space, without otherwise limiting the proof in any way, we will assume that the second group contains only two equations.

Therefore, let

$$(I)\quad \begin{array}{l}[av]+A=0\\ [bv]+B=0\\ [cv]+C=0\\ \quad\cdots\end{array} \qquad (II)\quad \begin{array}{l}[dv]+D=0\\ [ev]+E=0\end{array}$$

be the two sets of conditions that link the unknown corrections v. The first partial compensation based on the system (I) will be given by the formulas

$$(1)\quad \begin{array}{l}[aa]h_a+[ab]h_b+\cdots+A=0\\ [ab]h_a+[bb]h_b+\cdots+B=0\\ \quad\cdots\end{array}$$

$$(2)\quad \lambda'_r=a_rh_a+b_rh_b+\cdots \qquad (r=1,2,3,\ldots,n)$$

The corrections λ' thus found substituted in place of the letters v in the group (II) will not satisfy it in general, but will have residues D', E'. given by:

$$(3)\quad \begin{array}{l}[d\lambda']+D=D'\\ [e\lambda']+E=E'\end{array}$$

which, according to (2), yields

$$(4)\quad \begin{array}{l}[ad]h_a+[bd]h_b+\cdots+D=D'\\ [ae]h_a+[be]h_b+\cdots+E=E'\\ \quad\cdots\end{array}$$

The second partial compensation, according to the group (II) will be obtained with the formulas:

$$(5)\quad \begin{array}{l}[dd]h_d+[de]h_e+D'=0\\ [de]h_d+[ee]h_e+E'=0\end{array}$$

$$(6)\quad \lambda''_r=d_rh_d+e_rh_e.$$

[3] Die Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate. VIL Absch.

The newly corrected observations will now in general not satisfy system (I), and we will have the residuals

$$(7)\quad \begin{aligned} &[a\lambda'_r] + [a\lambda''_r] + A = A' \\ &[b\lambda'_r] + [b\lambda''_r] + B = B' \\ &\cdots \end{aligned}$$

or, taking into account (1) and (2)

$$(8)\quad \begin{aligned} &[a\lambda''_r] = A' \qquad &A' = [ad]h_d + [ae]h_e \\ &[b\lambda''_r] = B' \text{ i.e. } &B' = [bd]h_d + [be]h_e \\ &\cdots &\cdots \end{aligned}$$

A new compensation, according to the system (I) will therefore have to be carried out with the formulas:

$$(9)\quad \begin{aligned} &[aa]h'_a + [ab]h'_b + \cdots + A' = 0 \\ &[ab]h'_a + [bb]h'_b + \cdots + B' = 0 \end{aligned}$$

$$(10)\quad \mu'_r = a_r h'_a + b_r h'_b + \cdots$$

where the μ' denotes the new correction. Switching back to system (II) you will have, as you can easily see, the residuals:

$$(11)\quad \begin{aligned} D'' &= [ad]h'_a + [bd]h'_b + \cdots \\ E'' &= [ae]h'_a + [be]h'_b + \cdots \end{aligned}$$

and you will perform the new compensation, by means of the corrections μ'' provided by the formulas:

$$(12)\quad \begin{aligned} &[dd]h'_d + [de]h'_e + D'' = 0 \\ &[de]h'_d + [ee]h'_e + E'' = 0 \end{aligned}$$

$$(13)\quad \mu''_r = d_r h'_d + e_r h'_e.$$

The calculation will end when you get to a system quantity $h_d^{(s)}, h_e^{(s)}$ or $h_a^{(s)}, h_b^{(s)}, \ldots$ of negligible magnitude. Then the most likely final corrections will be given by the relations:

$$(14)\quad L_r = a_r k_a + b_r k_b + \cdots + d_r k_d + e_r k_e$$

where

$$(15)\quad \begin{aligned} k_a &= h_a + h'_a + \cdots + h_a^{(s)} \\ k_b &= h_b + h'_b + \cdots + h_b^{(s)}\ . \\ &\cdots \end{aligned}$$

In the practical execution of the calculation the determination of the partial corrections $\lambda', \lambda'', \mu', \mu''$ etc, is not needed, being sufficient the evaluation of the h's by means of the system of formulas (1) (4) (5) (9) (11) (12) etc.

The formulas now set forth make it manifest that the successive approximation procedure studied here is as fast as the smaller the summations

$$[ad], [bd], \ldots, [ae], [be], \ldots$$

are. It follows that when it comes to distributing by groups a certain number of condition equations, it will be more convenient to place in two different groups two given equations

$$[mv] + M = 0, \quad [pv] + P = 0,$$

that gives the smaller the summation $[mp]$e. This consideration can be of useful help to the calculator in the formation (in fact quite arbitrary) of groups of condition equations.

4. We will keep here exactly all the notations used in Note I, with respect to the two groups of equations considered there. We will also introduce the systems of quantities

$$\begin{array}{cccc} \delta_1, & \delta_2, & \ldots & \delta_n \\ \varepsilon_1, & \varepsilon_2, & \ldots & \varepsilon_n \end{array}$$

related to d, e by the relations

$$(16) \quad \begin{array}{ll} [dd][\delta\delta] + [de][\delta\varepsilon] = 1 & [dd][\delta\varepsilon] + [de][\varepsilon\varepsilon] = 0 \\ [de][\delta\delta] + [ee][\delta\varepsilon] = 0 & [de][\delta\varepsilon] + [ee][\varepsilon\varepsilon] = 1 \end{array}$$

$$(17) \quad \begin{array}{l} \delta_r = d_r[\delta\delta] + e_r[\delta\varepsilon] \\ \varepsilon_r = d_r[\delta\varepsilon] + e_r[\varepsilon\varepsilon] \end{array}$$

With the mentioned notations, the equations (9) of the previous paragraph solved with respect to $h'_a, h'_b, \ldots$ give:

$$\begin{array}{l} h'_a = -A'[\alpha\alpha] - B'[\alpha\beta] - \cdots \\ h'_b = -A'[\alpha\beta] - B'[\beta\beta] - \cdots \\ \quad\ldots \end{array}$$

or, by substituting for $A', B', \ldots$ their values given by (8)

$$(18) \quad \begin{array}{l} h'_a = -[\alpha d]h_d - [\alpha e]h_e - \cdots \\ h'_b = -[\beta d]h_d - [\beta e]h_e - \cdots \\ \quad\ldots \end{array}$$

Similarly, (11) (12) give, eliminating the D'', E'':

$$(19) \quad \begin{array}{l} h'_d = -[a\delta]h'_a - [b\delta]h'_b - \cdots \\ h'_e = -[a\varepsilon]h'_a - [b\varepsilon]h'_b - \cdots \end{array}$$

where, taking into account (17), we have denoted

$$(P) \quad [ad][\delta\delta] + [ae][\delta\varepsilon] = [a\delta]$$

and so on.

Finally, eliminating $h'_a, h'_b, \dots$ from (18) (19) we have:

$$(20)\qquad \begin{aligned} h'_d &= \{[a\delta][\alpha d] + [b\delta][\beta d] + \cdots\}h_d + \{[a\delta][\alpha e] + [b\delta][\beta e] + \cdots\}h_e, \\ h'_e &= \{[a\varepsilon][\alpha d] + [b\varepsilon][\beta d] + \cdots\}h_d + \{[a\varepsilon][\alpha e] + [b\varepsilon][\beta e] + \cdots\}h_e. \end{aligned}$$

Let us now consider the equations (10) of Note I and multiply the first of them by $[dd]$, the third by $[de]$, then add, taking into account (P), we have:

$$1 = [a\delta][\alpha d] + [b\delta][\beta d] + \cdots + [pp][\delta\delta] + [pq][\delta\varepsilon].$$

From this, and others similarly obtained, we have new very simple expressions of the coefficients of the relations (20), which can therefore be written:

$$\begin{aligned} h'_d &= \{1 - [pp][\delta\delta] + [pq][\delta\varepsilon]\}h_d - \{[pq][\delta\delta] + [qq][\delta\varepsilon]\}h_e, \\ h'_e &= -\{[pp][\delta\varepsilon] + [pq][\varepsilon\varepsilon]\}h_d + \{1 - [pq][\delta\varepsilon] - [qq][\varepsilon\varepsilon]\}h_e, \end{aligned}.$$

Multiplying the first of these by $[dd]$ and adding, then the first one by $[de]$, the second by $[ee]$ and adding again, we have:

$$(21)\qquad \begin{aligned} h'_d[dd] + h'_e[de] &= h_d\{[dd] - [pp]\} + h_e\{[de] - [pq]\}, \\ h'_d[de] + h'_e[ee] &= h_d\{[de] - [pq]\} + h_e\{[ee] - [qq]\}. \end{aligned}$$

Let us now define,

$$m_r = d_r - p_r, \quad n_r = e_r - q_r.$$

We have, taking into account the formulas (9) of Note I:

$$(22)\quad [mm] = [dd] - [pp],\ [mn] = [de] - [pq],\ [nn] = [ee] - [qq].$$

Thus, equation (21) can be written as follows:

$$(23)\qquad \begin{aligned} h'_d[pp] + h'_e[pq] &= (h_d - h'_d)[mm] + (h_e - h'_e)[mn], \\ h'_d[pq] + h'_e[qq] &= (h_d - h'_d)[mn] + (h_e - h'_e)[nn]. \end{aligned}$$

Multiplying the first of these by $(h_d - h'_d)$, the second by $(h_e - h'_e)$, and adding up we have:

$$\sum(ph_d + qh_e)(ph'_d + qh'_e) - \sum(ph'_d + qh'_e)^2 = \sum\{m(h_d - h'_d) + n(h_e - h'_e)\}^2.$$

Therefore, we have without difficulty

$$(25)\qquad \begin{aligned} &\textstyle\sum(ph_d + qh_e)^2 - \sum(ph'_d + qh'_e)^2 = \\ &= \textstyle\sum\{p(h_d - h'_d) + q(h_e - h'_e)\}^2 + 2\sum\{m(h_d - h'_d) + n(h_e - h'_e)\}^2. \end{aligned}$$

So, we have in any case:

$$\sum(ph_d + qh_e)^2 > \sum(ph'_d + qh'_e)^2.$$

It is worth noting that the difference between these two summations is consistently nonzero until h_d and h_e are both null and that therefore, by the law of continuity, such a difference cannot be arbitrarily small unless h_d, h_e are also arbitrarily small. In fact, the difference cannot be zero unless (see the note on p.* sect.*)

$$h_d - h'_d = 0, \quad h_e - h'_e = 0.$$

Now equation (21) can be written:

$$\begin{aligned}(h_d - h'_d)[dd] + (h_e - h'_e)[de] &= h_d[pp] + h_e[pq],\\ (h_d - h'_d)[de] + (h_e - h'_e)[ee] &= h_d[pq] + h_e[qq],\end{aligned}$$

which show that $h_d - h'_d$, $h_e - h'_e$, cannot be zero as long as h_d, h_e are different from zero. An exception would be the case in which the determinant

$$\begin{vmatrix} [pp] & [pq] \\ [pq] & [qq] \end{vmatrix}$$

is zero. But for the particular meaning of the letters p, q, this can not practically happen as we have shown in §3 of Note I.

The quantities h''_d, h''_e, which appear in a third round of compensation, are linked to h'_d, h'_e, by the same relationships that link the latter to h_d, h_e. Therefore, we will always have:

$$\sum(ph'_d + qh'_e)^2 > \sum(ph''_d + qh''_e)^2$$

and also

$$\sum(ph''_d + qh''_e)^2 > \sum(ph'''_d + qh'''_e)^2,$$

and so on.

5. Thus, it can be seen that, in successive rounds of compensation, i.e. as s increases, the function

$$F(s) = \sum(ph_d^{(s)} + qh_e^{(s)})^2$$

is continuously decreasing, and, since it cannot become negative, it must have a limit. But it is easy to see that this limit is zero. In fact, since the limit exists, the difference

$$F(s) - F(s+1)$$

can become arbitrarily small for s conveniently large. But for the observation made in § above, this requires that $F(s)$ also be arbitrarily small. The limit of $F(s)$ is therefore zero. It follows that as we proceed with the successive approximation calculation, that is as s increases, the binomials

$$\begin{gathered} p_1 h_d^{(s)} + q_1 h_e^{(s)} \\ p_2 h_d^{(s)} + q_2 h_e^{(s)} \\ \dots \\ p_n h_d^{(s)} + q_n h_e^{(s)} \end{gathered}$$

all tend to zero, which cannot happen unless, $h_d^{(s)}, h_e^{(s)}$ do not also tend to zero[4]. It is clear therefore that after a certain number of partial compensations the quantities $h_d^{(s)}, h_e^{(s)}$ will be reduced to a negligible size, and the general compensation can be considered complete.

[4]These binomials cannot be zero for values of $h_d^{(s)}, h_e^{(s)}$ both different from zero: in fact, if this were the case, we would have

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}$$

and for what has been observed in paragraph §3 of Note I the proposed condition equations would not be independent of each other. The said binomials cannot even be zero when one of the two quantities $h_d^{(s)}, h_e^{(s)}$ is zero and the other non-zero. In fact, if $h_d^{(s)} = 0, h_e^{(s)} \neq 0$, in order for those binomials to be zero there should be $q_1 = q_2 = \cdots = q_n = 0$. In this case for the second relation of (15 bis) of Note I there would still be a linear relation between the coefficients of the various condition equations, and these would not be, as we must suppose, independent of each other. For a similar reason the 2nd member of the formula (25) in the previous § cannot be zero unless we have $h_d - h'_d = 0$ and $h_e - h'_e = 0$.