About the characteristic roots of a linear substitution with an application to the theory of integral equations

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This a partial translation of Schur's paper.

Let the algebraic equation

$$\begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix} = 0$$

be given and $\omega_1, \ldots, \omega_n$ be its roots, as Mr. A. Hirsch[1] has shown, an upper limit for the absolute value of ω_ν can be specified in a very simple manner; namely, if a is the largest of the n^2 numbers $|a_{\chi,\lambda}|$, so

$$|\omega_\nu| \leq na.$$

In the present work, I would like to focus on a few other bounds and to draw attention to sharper inequalities that exist for the absolute values of ω_ν. The simplest and most important of them is the inequality

$$\sum_{\nu=1}^{n} |\omega_\nu|^2 \leq \sum_{\chi,\lambda} |a_{\chi\lambda}|^2.$$

By stating the meaning of the occurrence of the equal sign here, I get a result (Theorem II) that includes a number of known results on Hermitian forms and orthogonal substitutions as a special case.

The second section contains an application of my algebraic results on the theory of the linear homogeneous integral equation

$$\lambda \int_a^b K(s,t)\varphi(t)\,dt = \varphi(s).$$

[1] "Sur les racines d'une équation fondamentale", Acta. Mathematica, v. 25, 5. 367.

Considering a result of Mr. E. Schmidt for real symmetrical kernel $K(s,t)$, I show how the concept of the order of an eigenvalue λ can be justified elementarily for a general kernel without using Fredholm's transcendental entire function.

At the same time, it gives, under general conditions on the kernel $K(s,t)$, a proof of absolute convergence the series $\sum \frac{1}{\lambda^2}$, in which each eigenvalue λ is to be written as often as its order indicates.

Section I.

§1.

In the following, the complex conjugate of a is denoted as $\bar{a}$. Likewise, if $P = (p_{\chi\lambda})$ is a matrix (linear homogeneous substitution), $\bar{P}$ denotes a matrix with entries $\bar{p}_{\chi\lambda}$; P' denotes, as usual, the conjugate [transpose] of P.

P satisfies the equation

$$(1) \quad \bar{P}'P = E,$$

in which $E = (e_{\chi\lambda})$ is the identity matrix that is,

$$\sum_\nu \bar{p}_{\nu\chi} p_{\nu\lambda} = e_{\chi\lambda}, \quad (e_{\chi\chi} = 1,\, e_{12} = e_{13} = \cdots = 0)$$

this is what I call a unitary or a unitary orthogonal matrix P according to Mr. L. Autonne[2]. As is well known, the meaning of equation (1) is that the Hermitian unit form

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots$$

remains unchanged if x_χ is replaced by $\sum_\lambda p_{\chi\lambda} x_\lambda$ (and also $\bar{x}_\lambda$ is replaced by $\sum_\lambda \bar{p}_{\chi\lambda}\bar{x}_\lambda$). If the coefficients of P are real, P becomes an ordinary (real) orthogonal matrix.

Let

$$A = (a_{\chi\lambda}), \quad (\chi, \lambda = 1, 2, \ldots, n)$$

be a matrix with any real or complex coefficients. The characteristic roots of A, i.e. the roots of equation

$$(2) \qquad |A - xE| = \begin{vmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{vmatrix} = 0,$$

[2] "Sur 1'Hermitien", Rendiconti del Circolo Matematico di Palermo, T. 16 (1902), p. 104.

are $\omega_1, \ldots, \omega_n$. Then, we have the following result,

I. A unitary orthogonal matrix P can always be determined so that the matrix

$$\bar{P}'AP = P^{-1}AP = \mathsf{A}$$

takes the form

$$\mathsf{A} = \begin{vmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ c_{21} & \omega_2 & 0\cdots & 0 & \\ c_{31} & c_{32} & \omega_3 & \cdots & 0 \\ c_{n1} & c_{n2} & c_{n3} & \cdots & \omega_n \end{vmatrix}.$$

This is easy to prove[3].

Because ω_1 is a root of equation (2), we can determine n nonzero numbers $q_1, \ldots, q_n$ that satisfy the n equations

$$\sum_{\lambda=1}^{n} a_{\lambda\chi} q_\lambda = \omega_1 q_\chi, \quad (\chi = 1, 2, \ldots, n)$$

The sum $\sum_{\chi=1}^{N} \bar{q}_\chi q_\chi = q$ is then a positive number. If you denote

$$q_{\chi 1} = \frac{q_\chi}{\sqrt{q}},$$

then,

$$\sum_{\lambda=1}^{n} a_{\lambda\chi} q_{\lambda 1} = \omega_1 q_{\chi 1}$$

and

$$\sum_{\lambda} \bar{q}_{\lambda 1} q_{\lambda 1} = 1.$$

Now, in a known manner, determine $n(n-1)$ numbers

$$\begin{matrix} q_{12}, & q_{13}, & \cdots, & q_{1n} \\ q_{22}, & q_{23}, & \cdots, & q_{12n} \\ \vdots & \vdots & & \vdots \\ q_{n2}, & q_{n3}, & \cdots, & q_{nn} \end{matrix}$$

such that

$$\sum_{\nu=1}^{n} \bar{q}_{\nu\chi} q_{\nu\lambda} = e_{\chi\lambda}, \quad (\chi, \lambda = 1, 2, \ldots, n)$$

Then the matrix

$$Q = (q_{\chi\lambda})$$

[3] See L. Stickelberger, “Uber reelle orthogonale Substitutionen”, Programm der polyt. Schule, Zurich, 1877, and L. Autonne, op. cit., p. 119.

is a unitary matrix, and

$$Q'A(Q')^{-1} = \bar{Q}^{-1}A\bar{Q} = \begin{pmatrix} \omega_1 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix},$$

where $b_{\chi\lambda}$ stands for certain numbers. The characteristic roots of the matrix

$$B = \begin{pmatrix} b_{22} & b_{23} & \cdots & b_{2n} \\ b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

are $\omega_2, \ldots, \omega_n$. If we now assume that the theorem to be proved, which is evident for $n = 1$, is correct for matrices with $n-1$ rows and columns, a unitary matrix

$$R = \begin{pmatrix} r_{22} & r_{23} & \cdots & r_{2n} \\ r_{32} & r_{33} & \cdots & r_{3n} \\ \vdots & \vdots & & \vdots \\ r_{n2} & r_{n3} & \cdots & r_{nn} \end{pmatrix}$$

can be determined such that

$$R^{-1}BR = \begin{pmatrix} \omega_2 & 0 & \cdots & 0 \\ c_{32} & \omega_3 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{n2} & c_{n3} & \cdots & \omega_n \end{pmatrix}$$

If we then set

$$S = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & r_{n2} & \cdots & r_{nn} \end{pmatrix}$$

$P = \bar{Q}S$ becomes a unitary orthogonal matrix that satisfies the conditions of our theorem[4].

Theorem I can be rephrased as

I*. For each linear homogeneous substitution A in n variables $x_1, \ldots, x_n$, n linear forms $y_1, \ldots, y_n$ can be specified, such that the following conditions are satisfied:

1.

$$y_1\bar{y}_1 + y_2\bar{y}_2 + \cdots + y_n\bar{y}_n = x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n;$$

[4]If the $a_{\chi\lambda}$ and the ω_ν are real, then P can also be chosen as a real (orthogonal) matrix.

2. the linear form y_χ changes to a form that can be expressed linearly and homogeneously by $y_1, \ldots, y_\chi$ by applying the substitution A.

§2.

From

$$(3) \qquad \bar{P}' A P = \mathsf{A}$$

one has

$$(3') \qquad \bar{P}' \bar{A}' P = \bar{\mathsf{A}}'.$$

Because of $\bar{P}' P = E$

$$\bar{P}' A \bar{A}' P = \mathsf{A} \bar{\mathsf{A}}'.$$

But $\bar{P}' = P^{-1}$, hence the matrices $A\bar{A}'$ and $\mathsf{A}\bar{\mathsf{A}}'$ as similar matrices have the same trace, and since the trace of a matrix of the form $P\bar{P}'$ is, as you can easily see, nothing else than the sum of the squares of the absolute values of all coefficients of P, so we get

$$\sum_{\chi,\lambda} |a_{\chi\lambda}|^2 = \sum_{\nu} |\omega_\nu|^2 + \sum_{\chi>\lambda} |c_{\chi\lambda}|^2.$$

In particular, we have

$$\sum_{\nu} |\omega_\nu|^2 \leq \sum_{\chi,\lambda} |a_{\chi\lambda}|^2.$$

This is an equality if and only if the numbers $c_{\chi\lambda}$ are all zero. This gives us the result:

II. If $A = (a_{\chi\lambda})$ is any linear homogeneous substitution in n variables with the characteristic roots $\omega_1, \ldots, \omega_n$, then

$$\sum_{\nu} |\omega_\nu|^2 \leq \sum_{\chi,\lambda} |a_{\chi\lambda}|^2.$$

This is an equality if and only if A can be derived from the diagonal form

$$\Delta = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{pmatrix}$$

by a unitary orthogonal transformation of the variables.

At the same time there is the strange result that, as soon as

$$\sum_{\nu} |\omega_\nu|^2 = \sum_{\chi,\lambda} |a_{\chi\lambda}|^2$$

the determinant $|A - xE|$ must have all linear elementary divisors.

This result can be generalized somewhat.

If the elementary divisors of $|A - xE|$ are all linear, then a matrix Q of non-vanishing determinant can be determined, such that

$$Q^{-1}AQ = \Delta$$

Then,

$$\bar{Q}'A(\bar{Q}')^{-1} = \bar{\Delta}',$$

and

$$\Delta\bar{\Delta}' = Q^{-1}AQ\bar{Q}'A(\bar{Q}')^{-1}.$$

The trace $\sum_\nu |\omega_\nu|^2$ of $\Delta\bar{\Delta}'$ is therefore the same as the tracce of the right matrix or, which is the same, equal to the trace of the matrix

$$AQ\bar{Q}'A(\bar{Q}')^{-1}Q^{-1} = AH\bar{A}'H^{-1},$$

where $H = Q\bar{Q}'$. Here H is the matrix of a positive definite Hermitian form of non-vanishing determinant or, as one can say briefly, a positive Hermitian matrix.

Conversely, such a matrix H can be determined such that the trace of $AH\bar{A}'H^{-1}$ equals $\sum_\nu |\omega_\nu|^2$, and it is possible to choose a matrix Q for which $H = Q\bar{Q}'$. Then, $\sum_\nu |\omega_\nu|^2$ is also the trace of

$$Q^{-1}AQ\bar{Q}'A(\bar{Q}')^{-1}Q^{-1} = B\bar{B}',$$

where $B = Q^{-1}AQ$. The characteristic roots of the matrix $B = (b_{\chi\lambda})$, which is similar to A, are again $\omega_1, \ldots, \omega_n$, and

$$\sum_\nu |\omega_\nu|^2 = \sum_{\chi,\lambda} |b_{\chi\lambda}|^2$$

so B and therefore A are similar to the diagonal matrix Δ.

[...]