

On the convergence of Q-OR and Q-MR Krylov methods for solving linear systems

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Many **Krylov** methods have been proposed over the years for solving linear systems iteratively

Most of them can be classified as quasi-orthogonal (**Q-OR**) or quasi-minimum residual (**Q-MR**)

Q-OR: FOM, BiCG, Hessenberg, . . .

Q-MR: GMRES, truncated GMRES, QMR, CMRH, . . .

Whatever their definition, these methods share many fundamental properties

See the nice paper by [M. Eiermann and O.G. Ernst](#), *Geometric aspects in the theory of Krylov subspace methods*, Acta Numerica, v 10 n 10 (2001), pp. 251–312

These methods differ mainly by the basis of the [Krylov](#) space that is constructed:

- orthogonal for [FOM/GMRES](#) (true OR/MR methods)
- bi-orthogonal for [BiCG/QMR](#)
- based on an LU factorization of the Krylov matrix for [Hessenberg/CMRH](#)

Our aim is to show that some results about [GMRES](#) convergence can be extended to other [Q-OR/Q-MR](#) methods

GMRES

GMRES uses the Arnoldi process to construct an orthonormal basis of the Krylov subspace

$$\mathcal{K}_n(A, b) = \{b \quad Ab \quad \dots \quad A^{n-1}b\}$$

where n is the order of A . Assume the basis vectors are linearly independent. Then, V gives a basis of $\mathcal{K}_n(A, b)$ and

$$AV = VH, \quad V^*V = I,$$

the matrix H is (unreduced) upper Hessenberg

With $x_0 = 0$, the GMRES iterates $x_k = V_k y_k$ are computed by solving

$$\min_{x_k \in \mathcal{K}_k(A, b)} \|b - Ax_k\|$$

with V_k $n \times k$, k first columns of V

With $x_0 = 0$ we have

$$\|r_k\| = \|b - Ax_k\| = \min_{p \in \Pi_k} \|p(A)b\|$$

where Π_k is the set of polynomials of degree k with a value 1 at the origin

This leads to bounds for the residual norms. If A is diagonalizable, $A = X\Lambda X^{-1}$, it yields

$$\|r_k\| \leq \kappa(X) \min_{p \in \Pi_k} \|p(\Lambda)\| \|b\|$$

The main question is: What quantities does **GMRES** convergence depend on for a general non singular system?

What do we know about GMRES?

In exact arithmetic **GMRES** is a *direct* method. It terminates at iteration n

Let

$$K = (b \quad Ab \quad A^2b \quad \dots \quad A^{n-1}b)$$

be the **Krylov** matrix that we assume of full rank. Then

$$K = VU$$

with V orthogonal (or unitary) and U upper triangular with positive real diagonal entries

As we know, the matrix $H = V^*AV$ is upper Hessenberg

We have

$$H = UCU^{-1}$$

where C is the companion matrix for the eigenvalues of A

[This is a consequence of $AK = KC$]

Let x_k^G (resp. x_k^F) be the iterates for GMRES (resp. FOM) and the residual vectors $r_k^G = b - Ax_k^G$ (resp. $r_k^F = b - Ax_k^F$)

Without loss of generality we assume $x_0 = 0$ and $\|b\| = 1$

We know that

- every non-increasing residual norm convergence curve is possible for GMRES
- one can construct matrices A with a prescribed spectrum and right-hand sides b such that GMRES yields a prescribed decreasing residual norm convergence curve. In addition one can prescribe the Ritz values for all iterations
- we have two parametrizations of this class of matrices and right-hand sides

For these properties, see

A. Greenbaum and Z. Strakoš, *Matrices that generate the same Krylov residual spaces*, in Recent advances in iterative methods, G.H. Golub, A. Greenbaum and M. Luskin, eds., Springer, (1994), pp. 95–118

A. Greenbaum, V. Pták and Z. Strakoš, *Any nonincreasing convergence curve is possible for GMRES*, SIAM J. Matrix Anal. Appl., v 17 (1996), pp. 465–469

M. Arioli, V. Pták and Z. Strakoš, *Krylov sequences of maximal length and convergence of GMRES*, BIT Numerical Mathematics, v 38 n 4 (1998), pp. 636–643

J. Duintjer Tebbens and G. Meurant, *Any Ritz value behavior is possible for Arnoldi and for GMRES*, SIAM J. Matrix Anal. Appl., v 33 n 3 (2012), pp. 958–978

The APS parametrization

Greenbaum and Strakoš (1994) proved that any convergence curve for the residual norm that can be generated with GMRES can be obtained with a matrix having prescribed eigenvalues

Greenbaum, Pták and Strakoš (1996) showed that any nonincreasing sequence of residual norms can be given by GMRES

Arioli, Pták and Strakoš (1998) gave a complete parametrization of all pairs $\{A, b\}$ generating a prescribed residual norm convergence curve

Theorem (APS)

Assume we are given n positive numbers such that

$$1 = f_0 \geq f_1 \geq \cdots \geq f_{n-1} > 0$$

and n complex numbers $\lambda_1, \dots, \lambda_n$ all different from 0. Let A be a matrix of order n and b an n -dimensional vector. The following assertions are equivalent:

- 1- The spectrum of A is $\{\lambda_1, \dots, \lambda_n\}$ and GMRES applied to A and b yields residuals r_j^G , $j = 0, \dots, n-1$ such that

$$\|r_j^G\| = f_j, \quad j = 0, \dots, n-1$$

- 2- The matrix A is of the form $A = WYCY^{-1}W^*$ and $b = Wh$, where W is any unitary matrix, Y is given by

$$Y = \begin{pmatrix} h & R \\ & 0 \end{pmatrix}$$

R being any nonsingular upper triangular matrix of order $n - 1$, h a vector such that

$$h = (\eta_1, \dots, \eta_n)^T, \quad \eta_j = (f_{j-1}^2 - f_j^2)^{1/2}$$

and C is the companion matrix corresponding to the polynomial q ,

$$q(z) = (z - \lambda_1) \cdots (z - \lambda_n) = z^n + \sum_{j=0}^{n-1} \alpha_j z^j$$

$$C = \begin{pmatrix} 0 & \dots & & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & 0 & \dots & -\alpha_2 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -\alpha_{n-1} \end{pmatrix}$$

These results have led some people to write that “GMRES convergence for non-normal matrices does not depend on the eigenvalues”

We are going to see in a moment that this statement is not correct

It must be: “GMRES convergence for non-normal matrices does not depend **only** on the eigenvalues”

Another parametrization, GM-JDT

Assume we are given n positive numbers such that

$$1 = f_0 \geq f_1 \geq \cdots \geq f_{n-1} > 0$$

and n complex numbers $\lambda_1, \dots, \lambda_n$ all different from 0. Let A be a matrix of order n and b an n -dimensional vector. The following assertions are equivalent:

- 1- The spectrum of A is $\{\lambda_1, \dots, \lambda_n\}$ and GMRES applied to A and b yields residuals r_j^G , $j = 0, \dots, n-1$ such that

$$\|r_j^G\| = f_j, \quad j = 0, \dots, n-1$$

- 2- The matrix A is of the form $A = VUCU^{-1}V^*$ and $b = Ve_1$, where V is any unitary matrix, U is nonsingular upper triangular such that

$$U_{1,1}^{-1} = 1, \quad U_{1,j}^{-1} = \left(\frac{1}{f_{j-1}^2} - \frac{1}{f_{j-2}^2} \right)^{\frac{1}{2}}, \quad j = 2, \dots, n$$

and C is the companion matrix corresponding to the prescribed eigenvalues

This type of parametrization can also be used to prescribe all the Ritz values at every iteration

Moreover

- $|(U^{-1})_{1,k}| = 1/\|r_{k-1}^F\|$
- $\|r_k^G\|^2 = 1/(M_{k+1}^{-1})_{1,1}$ with $M_{k+1} = U_{k+1}^* U_{k+1}$

The latter result has been proved by several people: Stewart, Zitko, Ipsen, Liesen, Rozložník and Strakoš, and Sadok

Following ideas from H. Sadok, to compute $(M_{k+1}^{-1})_{1,1}$ we use two “simple” tools:

- ▶ Cramer’s rule (1750 but known before that)
- ▶ The Cauchy-Binet formula (1812) for $\det(AB)$ with A and B rectangular



G. Cramer
1704-1752



A.L. Cauchy
(1789-1857)



J. Binet
(1786-1856)

The residual norms for diagonalizable matrices

Let $A = X\Lambda X^{-1}$ and $c = X^{-1}b$. Then the Krylov matrix is

$$K = X \begin{pmatrix} c & \Lambda c & \dots & \Lambda^{n-1}c \end{pmatrix}$$

Therefore

$$M = K^*K = \begin{pmatrix} c & \Lambda c & \dots & \Lambda^{n-1}c \end{pmatrix}^* X^*X \begin{pmatrix} c & \Lambda c & \dots & \Lambda^{n-1}c \end{pmatrix}$$

and

$$M_{k+1} = \mathcal{V}_{k+1}^* D_c^* X^* X D_c \mathcal{V}_{k+1}$$

with D_c diagonal with c_i as diagonal entries and ...

$$\mathcal{V}_{k+1} = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^k \\ 1 & \lambda_2 & \cdots & \lambda_2^k \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^k \end{pmatrix}$$

an $n \times (k + 1)$ Vandermonde-like matrix

Using Cramer's rule, we have

$$\begin{aligned} (M_{k+1}^{-1})_{1,1} &= \frac{1}{\det(M_{k+1})} \det \begin{pmatrix} 1 & m_{1,2} & \cdots & m_{1,k} \\ 0 & & & \\ \vdots & & M_{2:k+1,2:k+1} & \\ 0 & & & \end{pmatrix} \\ &= \frac{\det(M_{2:k+1,2:k+1})}{\det(M_{k+1})} \end{aligned}$$

Let $F = XD_c \mathcal{V}_{k+1}$, an $n \times (k+1)$ matrix. Then $M_{k+1} = F^* F$ and by the Cauchy-Binet formula

$$\det(M_{k+1}) = \sum_{I_{k+1}} |\det(F_{I_{k+1},:})|^2$$

the sum is over all sets I_{k+1} of $k+1$ row indices $(i_1, i_2, \dots, i_{k+1})$ such that $1 \leq i_1 < \dots < i_{k+1} \leq n$ and $F_{I_{k+1},:}$ is the submatrix of F whose row indices belong to I_{k+1}

$$F_{I_{k+1},:} = (XD_c)_{I_{k+1},:} \mathcal{V}_{k+1}, \quad [(k+1) \times n] * [n \times (k+1)]$$

Hence we can again apply the Cauchy-Binet formula

$$\det(F_{I_{k+1},:}) = \sum_{J_{k+1}} \det(X_{I_{k+1},J_{k+1}}) c_{j_1} \cdots c_{j_{k+1}} \det(\mathcal{V}(\lambda_{j_1}, \dots, \lambda_{j_{k+1}}))$$

with the **Vandermonde** matrix

$$\mathcal{V}(\lambda_{j_1}, \dots, \lambda_{j_{k+1}}) = \begin{pmatrix} 1 & \lambda_{j_1} & \cdots & \lambda_{j_1}^k \\ 1 & \lambda_{j_2} & \cdots & \lambda_{j_2}^k \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{j_{k+1}} & \cdots & \lambda_{j_{k+1}}^k \end{pmatrix}$$

But $\det(\mathcal{V}(\lambda_{j_1}, \dots, \lambda_{j_{k+1}}))$ is known

We apply the same technique for computing $\det(M_{2:k+1,2:k+1})$

The residual norms

Let A be a diagonalizable matrix with $A = X\Lambda X^{-1}$, $c = X^{-1}b$.

Then

$$\|r_k^G\|^2 = \sigma_{k+1}^N / \sigma_k^D$$

with

$$\sigma_{k+1}^N = \sum_{I_{k+1}} \left| \sum_{J_{k+1}} \det(X_{I_{k+1}, J_{k+1}}) c_{j_1} \cdots c_{j_{k+1}} \prod_{j_1 \leq j_l < j_p \leq j_{k+1}} (\lambda_{j_p} - \lambda_{j_l}) \right|^2$$

$$\sigma_1^D = \sum_{i=1}^n \left| \sum_{j=1}^n X_{i,j} c_j \lambda_j \right|^2$$

and

$$\sigma_k^D = \sum_{I_k} \left| \sum_{J_k} \det(X_{I_k, J_k}) c_{j_1} \cdots c_{j_k} \lambda_{j_1} \cdots \lambda_{j_k} \prod_{j_1 \leq j_l < j_p \leq j_k} (\lambda_{j_p} - \lambda_{j_l}) \right|^2, \quad k > 1$$

where the summations are over all sets of indices $I_{k+1}, J_{k+1}, I_k, J_k$ defined as I_ℓ to be a set of ℓ indices $(i_1, i_2, \dots, i_\ell)$ such that $1 \leq i_1 < \dots < i_\ell \leq n$, X_{I_ℓ, J_ℓ} is the submatrix of X whose row and column indices are defined by I_ℓ and J_ℓ

If the matrix A is normal, we have $X^*X = I$ and simpler formulas

$$\sigma_{k+1}^N = \sum_{l_{k+1}} |c_{j_1}|^2 \cdots |c_{j_{k+1}}|^2 \prod_{j_1 \leq j_i < j_p \leq j_{k+1}} |(\lambda_{j_p} - \lambda_{j_i})|^2$$

$$\sigma_1^D = \sum_{i=1}^n |c_j|^2 |\lambda_j|^2$$

and

$$\sigma_k^D = \sum_{l_k} |c_{j_1}|^2 \cdots |c_{j_k}|^2 |\lambda_{j_1}|^2 \cdots |\lambda_{j_k}|^2 \prod_{j_1 \leq j_i < j_p \leq j_k} |(\lambda_{j_p} - \lambda_{j_i})|^2, \quad k > 1$$

See the paper by J. Duintjer Tebbens, GM, H. Sadok and Z. Strakoš, LAA v 450 (2014)

The results for diagonalizable matrices can somehow be extended to the case of non-diagonalizable matrices using the [Jordan canonical form](#)

In particular we can obtain nice expressions of the residual norms for one Jordan block

Bounds for the residual norms (diagonalizable matrices)

Using a result by [Bellalij](#), [Jbilou](#) and [Sadok](#) we can prove that

$$\|r_k\|^2 \leq \frac{\|X\|^2}{\mathbf{e}_1^T (\mathcal{V}_{k+1}^* D_c^* D_c \mathcal{V}_{k+1}^*)^{-1} \mathbf{e}_1}$$

It yields

$$\|r_k\|^2 \leq \|X\|^2 \frac{\sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} \prod_{j=1}^{k+1} \omega_{i_j} \prod_{i_1 \leq i_\ell < i_p \leq i_{k+1}} |\lambda_{i_p} - \lambda_{i_\ell}|^2}{\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \omega_{i_j} |\lambda_{i_j}| \prod_{i_1 \leq i_\ell < i_p \leq i_k} |\lambda_{i_p} - \lambda_{i_\ell}|^2}$$

where $\omega_j = |c_j|^2$ with $c = X^{-1}b$

We can also obtain a similar lower bound where the multiplying factor is $\sigma_{\min}(X)^2$ (min of the singular values)

Hence, the residual norms depend on the (differences of the) eigenvalues, the eigenvectors and the right-hand side (through $X^{-1}b$)

If $\sigma_{\min}(X) \approx \|X\|$, the convergence depends essentially on the eigenvalues (and the right-hand side)

Q-OR and Q-MR methods

Can we extend some of these results to Q-OR and Q-MR methods?

We assume that we have an ascending basis V of the Krylov space (with columns of unit norm) such that $K = VU$ with V nonsingular and U upper triangular

We define $H = UCU^{-1}$. As a consequence $AV = VH$. The iterates are

$$x_k = V_k y_k$$

where V_k is the matrix of the k first columns of V . The residual r_k is

$$V_k e_1 - AV_k y_k = V_k (e_1 - H_k y_k) - h_{k+1,k} (y_k)_k v_{k+1} = V_{k+1} (e_1 - \underline{H}_k y_k)$$

The Q-OR method is defined (provided that H_k is nonsingular) by

$$H_k y_k^O = e_1$$

where H_k is the principal submatrix of order k . This annihilates the first term in the residual

In the Q-MR method y_k^M is computed as the solution of the least squares problem

$$\min_y \|e_1 - \underline{H}_k y\|$$

where \underline{H}_k is $(k+1) \times k$

The vector $z_k^M = e_1 - \underline{H}_k y_k^M$ is referred as the quasi-residual

The residual vector is $r_k^M = V_{k+1} z_k^M$

Generally, the two problems are solved using **Givens** rotations with sines s_j . It is known that

$$\|z_k^M\| = |s_1 s_2 \cdots s_k|$$

Moreover we have a relation between the **Q-OR** residual norms and the **Q-MR** quasi-residual norms

$$\frac{1}{\|r_k^O\|^2} = \frac{1}{\|z_k^M\|^2} - \frac{1}{\|z_{k-1}^M\|^2}$$

See Eiermann and Ernst (2001), Freund and Nachtigal (1991)

Properties of Q-OR and Q-MR methods

From these results we can show that

$$|(U^{-1})_{1,k}| = \frac{1}{\|r_{k-1}^O\|}$$

This is proved by using the rotation matrices G_j such that

$$G_{n-1} \cdots G_1 H = \mathcal{R}$$

From $H = UC U^{-1}$ we have

$$U^{-1} G_1^{-1} \cdots G_{n-1}^{-1} = C U^{-1} \mathcal{R}^{-1}$$

The first row of the matrix on the right-hand side is zero except for the last entry

Looking at the product of the inverses of the rotation matrices, we can show by induction that

$$U_{1,j+1}^{-1} = \frac{\tilde{c}_j}{s_1 \cdots s_j} = \pm \left(\frac{1}{(s_1 \cdots s_j)^2} - \frac{1}{(s_1 \cdots s_{j-1})^2} \right)^{1/2}, \quad j = 1, \dots, n-1$$

Let $M_{k+1} = U_{k+1}^* U_{k+1}$. Since $M_{k+1}^{-1} = U_{k+1}^{-1} U_{k+1}^{-*}$ and from the first row of U^{-1} , a consequence of the previous result is the following

$$\|z_k^M\|^2 = \frac{1}{(M_{k+1}^{-1})_{1,1}}$$

The difference with **GMRES** is that we only have the norm of the quasi-residual

Then, we obtain expressions of the quasi-residual norms

The quasi-residual norms

Let A be a diagonalizable matrix with $A = X\Lambda X^{-1}$, $Z = V^{-1}X$ and $c = X^{-1}b$. Then

$$\|z_k^M\|^2 = \sigma_{k+1}^N / \sigma_k^D$$

with

$$\sigma_{k+1}^N = \sum_{I_{k+1}} \left| \sum_{J_{k+1}} \det(Z_{I_{k+1}, J_{k+1}}) c_{j_1} \cdots c_{j_{k+1}} \prod_{j_1 \leq j_l < j_p \leq j_{k+1}} (\lambda_{j_p} - \lambda_{j_l}) \right|^2$$

$$\sigma_1^D = \sum_{i=1}^n \left| \sum_{j=1}^n Z_{i,j} c_j \lambda_j \right|^2$$

and

$$\sigma_k^D = \sum_{I_k} \left| \sum_{J_k} \det(Z_{I_k, J_k}) c_{j_1} \cdots c_{j_k} \lambda_{j_1} \cdots \lambda_{j_k} \prod_{j_1 \leq j_l < j_p \leq j_k} (\lambda_{j_p} - \lambda_{j_l}) \right|^2, \quad k > 1$$

where the summations are over all sets of indices $I_{k+1}, J_{k+1}, I_k, J_k$ where I_ℓ is a set of ℓ indices $(i_1, i_2, \dots, i_\ell)$ such that $1 \leq i_1 < \dots < i_\ell \leq n$, Z_{I_ℓ, J_ℓ} is the submatrix of Z whose row and column indices are defined by I_ℓ and J_ℓ

This result arises from $\|z_k^M\|^2 = 1/(M_{k+1}^{-1})_{1,1}$ and

$$\begin{aligned} M &= U^*U = K^*V^{-*}V^{-1}K \\ &= (c \ \Lambda c \ \dots \ \Lambda^{n-1}c)^* Z^*Z (c \ \Lambda c \ \dots \ \Lambda^{n-1}c) \end{aligned}$$

It yields

$$M_{k+1} = \mathcal{V}_{k+1}^* D_c^* Z^* Z D_c \mathcal{V}_{k+1}$$

where D_c is diagonal and \mathcal{V}_{k+1} is an $n \times (k+1)$ Vandermonde-like matrix

As for [GMRES](#), we compute the $(1, 1)$ entry of the inverse using [Cramer's](#) rule and the [Cauchy-Binet](#) determinant formula

Note that there is no simplification when A is normal ($Z^*Z \neq I$)

Construction of linear systems with a prescribed convergence curve

Can we construct linear systems with a prescribed convergence curve and a prescribed spectrum for Q-OR and Q-MR methods?

For FOM/GMRES this is easy since we just have to construct an upper triangular matrix U^{-1} with the inverses of the FOM residual norms (obtained from the GMRES norms) on the first row. Then we take

$$A = VUCU^{-1}V^*, \quad b = Ve_1$$

where C is the companion matrix of the given eigenvalues and V is any unitary matrix

Things are more difficult for some Q-OR/Q-MR methods because we may ask for some non-zero structure in H

BiCG

We would like to find matrices H tridiagonal (with a given spectrum) and U upper triangular such that

$$H = \begin{pmatrix} \gamma_1 & \beta_2 & 0 & 0 & 0 \\ \rho_2 & \gamma_2 & \beta_3 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \rho_{n-1} & \gamma_{n-1} & \beta_n \\ 0 & 0 & 0 & \rho_n & \gamma_n \end{pmatrix} = UCU^{-1}$$

and the first row of U^{-1} is prescribed as $(1 \ g_1 \ \cdots \ g_{n-1})$ with $g_j \neq 0$

Let $\omega_2, \dots, \omega_n$ be arbitrary chosen entries of the last column of U^{-1} with $\omega_n \neq 0$, $\omega_1 = g_{n-1}$ and $-\alpha_0, \dots, -\alpha_{n-1}$ be the entries of the last column of C that we know from the prescribed spectrum

We compute U^{-1} and H recursively column-wise
The last column of $U^{-1}H = CU^{-1}$ yields

$$\begin{pmatrix} g_{n-2}\beta_n + g_{n-1}\gamma_n \\ \vdots \\ \omega_n\gamma_n \end{pmatrix} = \begin{pmatrix} -\alpha_0\omega_n \\ \vdots \\ \omega_{n-1} - \alpha_{n-1}\omega_n \end{pmatrix}$$

We use the first and last equations

$$\begin{pmatrix} g_{n-2} & g_{n-1} \\ 0 & \omega_n \end{pmatrix} \begin{pmatrix} \beta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} -\alpha_0\omega_n \\ \omega_{n-1} - \alpha_{n-1}\omega_n \end{pmatrix}$$

The solution of this 2×2 non singular linear system yields γ_n, β_n

From the other equations that we discarded we can compute the unknown entries $\nu_{j,n-1}$ of column $n - 1$ of U^{-1}

Then we go on backwards with column $n - 1$

We have three unknowns $\beta_{n-1}, \gamma_{n-1}$ and ρ_n

We first take the first and the last two equations

This gives us a linear system with an upper triangular matrix

$$\begin{pmatrix} g_{n-3} & g_{n-2} & g_{n-1} \\ 0 & \nu_{n-1,n-1} & \omega_{n-1} \\ 0 & 0 & \omega_n \end{pmatrix} \begin{pmatrix} \beta_{n-1} \\ \gamma_{n-1} \\ \rho_n \end{pmatrix} = \begin{pmatrix} 0 \\ \nu_{n-2,n-1} \\ \nu_{n-1,n-1} \end{pmatrix}$$

And so on... Then $A = VHV^{-1}$ and $b = Ve_1$ for an appropriately chosen matrix V

So far we don't know how to completely handle the case with zero entries on the first row of U^{-1}

This algorithm can be easily extended to a larger upper bandwidth but what about stability?

This allows to prescribe BiCG (finite) residual norm convergence (or QMR quasi-residual norms)

Summary

- ▶ Many known properties of **FOM/GMRES** are also valid for general **Q-OR/Q-MR** methods
- ▶ We express the **Q-MR** quasi-residual norms as functions of the eigenvalues, the eigenvectors, the right-hand side and the basis of the **Krylov** space
- ▶ We (almost) have a parametrization of the class of matrices with a prescribed spectrum and a prescribed **Q-OR/Q-MR** convergence curve
- ▶ In particular we can construct examples with a **BiCG** (finite) convergence curve