On the algebraic analog of a theorem by Fejér

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Mr. Fejer[1] has proved the following theorem:

If $f(t)$, $g(t)$ are a pair of real continuous functions of the real variables t with period 2π, $f_n(t), g_n(t)$ being the n-th terms of their Fourier expansions, and furthermore let $\mathfrak{R}$ be the smallest convex region which contains the curve given by the parameter representation $x = f(t), y = g(t)$ completely, $\mathfrak{R}_n$, the corresponding region for the curve $x = f_n(t), y = g_n(t)$, then $\mathfrak{R}_n$, is contained in $\mathfrak{R}$.

In the following an analogue of this proposition from algebra will be obtained. To set up such an analogue, I was guided by the same principle that I used in my habilitation thesis[2] to establish a connection between the theory of Fourier series and a certain class of bilinear forms. But while this transfer principle, like in the theory of infinitely many variables in general, has so far mostly served to make certain tools of algebra usable for the problems of analysis, I want to show here with a simple example how one can also use this principle in the opposite way in order, starting from theorems in analysis, to arrive at algebraic facts.

This work does not assume knowledge of the theory of infinitely many variables, with the exception of the last paragraph; Incidentally, it only deals with bilinear forms with n pairs of variables.

§1.

The theorem of Bendixson and Hirsch.

It is known that the eigenvalues of a real quadratic form $\sum a_{\alpha,\beta} x_\alpha x_\beta$, i.e. the roots of the equation

$$\begin{vmatrix} a_{1,1}-\lambda & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2}-\lambda & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}-\lambda \end{vmatrix} = 0, \quad (a_{\alpha,\beta} = a_{\beta,\alpha})$$

[1] Über gewisse durch die Fouriersche und Laplacesche Reihe definierten Mittelkurven und Mittelflächen, Rendiconti del Circolo Matematico di Palermo 38 (1914), S. 79-97

[2] Math. Annalen 70 (1911), p. 351-376; Gött. Nachr. 1910, math.-phys. Kl. p. 489-506; Rendiconti del Circolo Matematico di Palermo 32 (1911), p. 191-192.

are all real, and that those of an alternating (skew symmetric, $a_{\alpha,\beta} + a_{\beta,\alpha} = 0$) bilinear form are all purely imaginary. These two facts can be subsumed into a single one if one uses the concept of "Hermitian form". A bilinear form

$$C = C(x, y) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} c_{\alpha,\beta} x_\alpha y_\beta$$

is called an Hermitian form if it is identical with its "accompanying" form[3]

$$\bar{C}' = \bar{C}'(x, y) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \bar{c}_{\beta,\alpha} x_\alpha y_\beta$$

and $c_{\alpha,\beta} = \bar{c}_{\beta,\alpha}$. If $c_{\alpha,\beta} = a_{\alpha,\beta} + i\, b_{\alpha,\beta}$, then

$$\begin{aligned} C &= \sum (a_{\alpha,\beta} + i\, b_{\alpha,\beta}) x_\alpha y_\beta = S + i\, T \\ \bar{C}' &= \sum (a_{\beta,\alpha} - i\, b_{,\beta,\alpha}) x_\alpha y_\beta = S' - i\, T', \end{aligned}$$

and from $C = \bar{C}'$ it follows by comparing the real and the imaginary parts $S = S'$, $T = -T'$, i.e. every Hermitian form is of the form $S + i\, T$, where S is real symmetric and T is a real alternating form. Thus the

Theorem 1: The eigenvalues of an Hermitian form are all real,

with the two facts mentioned at the beginning as special cases, depending on the choice of $T = 0$ or $S = 0$.

Mr. Bendixson[4] has established a generalization of this for arbitrary bilinear forms with real coefficients, and Mr. Hirsch[5] has remarked that it carries over directly to bilinear forms with complex coefficients. Let me begin with the remark:

Theorem 2. Every bilinear form C can be represented in one and only one way in the form $C = H + i\, K$, where H, K are Hermitian forms.

In fact, one only needs to put

$$H = \frac{C + \bar{C}'}{2}, \ K = \frac{C - \bar{C}'}{2i}$$

to calculate $C = H + i\, K$, and that H and K are Hermitian forms. But if C could be put in a second way into the form $C = H_1 + i\, K_1$, then

$$(H - H_1) + i\, (K - K_1) = 0,$$

and by transition to the complex conjugate and interchange of the two variables the simultaneous existence of the other relation

$$(H - H_1) - i\, (K - K_1) = 0$$

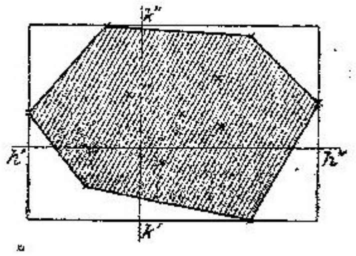
[3] The overbar is to indicate the conjugate and the prime the interchange of the two sets of variables in a bilinear form.

[4] Acta mathematica 25 (1902), p. 359-366.

[5] Acta mathematica 25 (1902), p. 367-370.

would follow; the addition and subtraction of both relations thus would give $H_1 = H$, $K_1 = K$, i.e. the uniqueness of that representation.

Theorem 3 (Theorem of Bendixson and Hirsch). If C is any bilinear form, H, K are its two Hermitians uniquely determined by Theorem 2 and if all eigenvalues of H, which are real by Theorem 1, are between h' and h'', all of K between k' and k'', then construct the rectangle parallel to the coordinate axes whose vertical sides have the abscissas h' and h'', the horizontal sides have the ordinates k' and k''. Then all eigenvalues of C lie within this rectangle.



It is clear that Theorem 3 reduces to Theorem 1 if C is an Hermitian form, i.e. $K = 0$, and the rectangle reduces to an interval of the real axis.

The proof follows very simply from the well-known remark, that the values given by an Hermitian form $H(x, y)$ for complex conjugate values of the two series of variables ($y_\alpha = \bar{x}_\alpha$) and under the constraint $x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = 1$, and which are all real, fill exactly the distance between their lowest and their highest eigenvalue. If λ is some eigenvalue of C, then one can - that is the definition of the eigenvalue - determine n numbers $x_1, \ldots, x_n$, so that

$$\sum_{\beta=1}^{n} c_{\alpha,\beta}\, x_\beta = \lambda x_\alpha \quad (\alpha = 1, \ldots, n)$$

and since here only the ratios of the quantities $x_1, \ldots, x_n$ are involved, one can still dispose of their proportionality factor in such a way that at the same time $x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = 1$. Multiplying the αth equation by $\bar{x}_\alpha$ and adding up over α, we have

$$\sum_{\alpha,\beta} c_{\alpha,\beta}\, \bar{x}_\alpha x_\beta = \lambda \sum_{\alpha} \bar{x}_\alpha x_\alpha = \lambda,$$

and therefore $\lambda = C(\bar{x}, x) = H(\bar{x}, x) + i\, K(\bar{x}, x)$, while $x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = 1$. So the real part of λ lies between the outermost eigenvalues h', h'' of H, the imaginary part between those k', k'' of K, i.e. λ lies in the rectangle drawn in the figure.

I have reproduced the proof here, because in this short form, it immediately shows that one can give an extension of Theorem 3 which is essential for the following. For this one needs the definition:

Definition. The field of values $\mathfrak{W}$ of a bilinear form $C(x, y)$ is to be understood as the totality of real and complex values which the expression

$$C(x, \bar{x}) = \sum_{\alpha,\beta} c_{\alpha,\beta}\, x_\alpha \bar{x}_\beta$$

assumes under the constraint $x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = 1$.

The field of value of an Hermitian form is a part of the real axis, and Theorem 3 can be extended directly as follows on the basis of the proof given above:

Theorem 4. The eigenvalues of an arbitrary bilinear form C all belong to its field of values $\mathfrak{W}$.

§2.
The extension of Fejér's theorem

Definition. A bilinear form $C(x, y)$ is called normal if it is is interchangeable with its accompanying form $\bar{C}'$, i.e. if

$$\sum_{p=1}^{n} c_{\alpha,p}\bar{c}_{\beta,p} = \sum_{p=1}^{n} c_{p,\alpha}\bar{c}_{p,\beta}, \quad (\alpha, \beta = 1, \ldots, n)$$

holds.

Theorem 5. Let $\mathfrak{R}$ be the smallest convex domain enclosing all eigenvalues of the bilinear form C, $\mathfrak{W}$ be the field of values of C. Then, if C is normal, $\mathfrak{W}$ is identical with $\mathfrak{R}$.

The proof results in a very simple way from the following theorem of Mr. I. Schur[6]:

An arbitrary bilinear form $C(x, y)$ can be transformed by a "unitary" transformation of the variables, i.e., by a transformation

$$x_\alpha = \sum_{\beta=1}^{n} w_{\alpha,\beta}\, \xi_\beta, \quad y_\alpha = \sum_{\beta=1}^{n} w_{\beta,\alpha}\, \eta_\beta$$

for which

$$\sum_{p=1}^{n} w_{\alpha,p}\bar{w}_{\beta,p} = e_{\alpha,\beta}, \quad \sum_{p=1}^{n} w_{p,\alpha}\bar{w}_{p,\beta} = e_{\alpha,\beta}$$

[6]Math. Annalen 66 (1909), p. 488-510, 3 1.

i.e., $= 0$ for $\alpha \neq \beta$, $= 1$ for $\alpha = \beta$, into another bilinear form $\Gamma(\xi, \eta)$ whose coefficient matrix is as follows:

$$\begin{vmatrix} \lambda_1 & 0 & \cdots & 0 \\ \gamma_{2,1} & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n,1} & \gamma_{n,2} & \cdots & \lambda_n \end{vmatrix}.$$

The values $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of C (and of Γ at the same time).

If C is normal, then - most conveniently with the help of the matrix calculus - it can be seen immediately that also the transformed form Γ is normal. Because from

$$\Gamma = \bar{W}'CW, \quad W\bar{X}' = \bar{X}'W = E$$

it follows

$$\begin{aligned} \Gamma\bar{\Gamma}' &= (\bar{W}'CW)\,(\bar{W}'\bar{C}'W) = \bar{W}'C\bar{C}'W, \\ \bar{\Gamma}'\Gamma &= (\bar{W}'\bar{C}'W)\,(\bar{W}'CW) = \bar{W}'\bar{C}'CW; \end{aligned}$$

since $C\bar{C}' = \bar{C}'C$ (i.e. C is normal), so also $\Gamma\bar{\Gamma}' = \bar{\Gamma}'\Gamma$.

From the special nature of the coefficient matrix of Γ it follows that Γ can be normal only if all $\gamma_{\alpha,\beta}$ below the diagonal are 0. For if Γ is normal, first the sum of squares of the elements of the 1st row is equal to the sum of squares of the elements of the 1st column, i.e.:

$$\lambda_1\bar{\lambda}_1 = \lambda_1\bar{\lambda}_1 + \gamma_{2,1}\bar{\gamma}_{2,1} + \cdots + \gamma_{n,1}\bar{\gamma}_{n,1},$$

and since these are all real, positive summands, $\gamma_{2,1} = 0, \ldots, \gamma_{n,1} = 0$. From the corresponding consideration for the 2nd line and 2nd column one concludes then $\gamma_{3,2} = 0, \ldots, \gamma_{n,2} = 0$ and so on. Thus one obtains

Conclusion from the theorem of I. Schur[7]. A bilinear form is unitary similar to a diagonal form, i.e. to a bilinear form of the shape

$$\Gamma(x, y) = \lambda_1 x_1 y_1 + \cdots + \lambda_n x_n y_n$$

if it is normal.

From this conclusion the proof of Theorem 5 indeed follows now in a very simple way. For the form $\Gamma(x, y)$, which is derived from $C(x, y)$ by a unitary transformation, obviously has the same set of values $\mathfrak{W}$ and also the

[7]This immediate conclusion, which Mr. I. Schur, loc. cit. did not express, was known to him as well as to me for a long time. I use this opportunity to supplement in this way a remark of Mr. A. Ostrowski [Math. Annalen 78 (1917), p. 118], who uses the "inference" and refers to me alone.

same eigenvalues $\lambda_1, \ldots, \lambda_n$ as C. If C is normal, then, according to Schur's theorem,

$$\Gamma(x, \bar{x}) = \lambda_1 x_1 \bar{x}_1 + \cdots + \lambda_n x_n \bar{x}_n$$

it can be seen immediately from this simple expression that under the restriction $x_1\bar{x}_1 + \cdots + x_n\bar{x}_n = 1$ all those complex values fill the smallest convex polygon containing $\lambda_1, \ldots, \lambda_n$ (cf. the above figure, in which this area is hatched).

§3.

The relationship between the field of value and the maximum of a bilinear form

Far more common than the previously considered field of value $\mathfrak{W}$[8] of a bilinear form $C = \sum c_{\alpha,\beta}\, x_\alpha y_\beta$ is another one: the field of value $\mathfrak{M}$ of the bilinear form, in which $y_\alpha = \bar{x}_\alpha$ are not set like this, but vary independently from each other and are only bound to the two constraints $\sum x_\alpha \bar{x}_\alpha = 1$ and $\sum y_\alpha \bar{y}_\alpha = 1$. Obviously, $\mathfrak{W}$ is contained in $\mathfrak{M}$. As one can easily see, $\mathfrak{M}$ is a circle around O; its radius M is what is commonly referred to as the "maximum of the bilinear form". On the other hand, if W is the radius of the smallest circle around O that contains $\mathfrak{W}$ entirely, then obviously $W \leq M$. The relationship between W and M is clarified by the following two theorems.

Theorem 6. $W \geq \frac{1}{2}M$.

Proof[9].

$$\begin{aligned}
C(x+y, \bar{x}+\bar{y}) &= C(x,\bar{x}) + C(y,\bar{y}) + C(x,\bar{y}) + C(y,\bar{x}), \\
C(x+i\,y, \bar{x}-i\,\bar{y}) &= C(x,\bar{x}) + C(y,\bar{y}) - i\,C(x,\bar{y}) + i\,C(y,\bar{x}), \\
C(x-y, \bar{x}-\bar{y}) &= C(x,\bar{x}) + C(y,\bar{y}) - C(x,\bar{y}) - C(y,\bar{x}), \\
C(x-i\,y, \bar{x}+i\,\bar{y}) &= C(x,\bar{x}) + C(y,\bar{y}) + i\,C(x,\bar{y}) - i\,C(y,\bar{x}).
\end{aligned}$$

Multiplying respectively by $1/4$, $i/4$, $-1/4$ and $-i/4$, and adding

$$\begin{aligned}
\frac{1}{4}C(x+y, \bar{x}+\bar{y}) &+ \frac{1}{4}iC(x+i\,y, \bar{x}-i\,\bar{y}) - \frac{1}{4}C(x-y, \bar{x}-\bar{y}) \\
&- \frac{1}{4}iC(x-i\,y, \bar{x}+i\,\bar{y}) = C(x,\bar{y}).
\end{aligned}$$

Now, from the fact that W is the maximum of $|C(x,\bar{x})|$ under the constraint $\sum x_\alpha \bar{x}_\alpha = 1$, we can immediately conclude that $|C(x,\bar{x})| \leq W \sum x_\alpha \bar{x}_\alpha$ holds for any x_α. So for any x_α, y_α:

$$\begin{aligned}
|C(x,\bar{y})| &\leq \frac{1}{4}W[\sum(x_\alpha + y_\alpha)\,(\bar{x}_\alpha + \bar{y}_\alpha) + \sum(x_\alpha + i\,y_\alpha)\,(\bar{x}_\alpha - i\,\bar{y}_\alpha) \\
&+ \sum(x_\alpha - y_\alpha)\,(\bar{x}_\alpha - \bar{y}_\alpha) + \sum(x_\alpha - i\,y_\alpha)\,(\bar{x}_\alpha + i\,\bar{y}_\alpha)] \\
&\leq \frac{1}{4}W[4\sum x_\alpha \bar{x}_\alpha + 4\sum y_\alpha \bar{y}_\alpha] = W[\sum x_\alpha \bar{x}_\alpha + \sum y_\alpha \bar{y}_\alpha].
\end{aligned}$$

[8] The definition of this term can be found at the end of §1 of this paper.

[9] The proof is modeled on a conclusion by E. Hellinger, Math. Annalen 69 (1909), p. 303

If we now choose x_α, y_α in such a way that $C(x, \bar{y})$ just reaches its maximum M of these values while $\sum x_\alpha \bar{x}_\alpha = 1$, $\sum y_\alpha \bar{y}_\alpha = 1$, then we get in fact $M \leq 2W$, thus $W \geq \frac{1}{2}M$.

I now consider the example of the special bilinear form $x_1 y_2$, of 2 pairs of variables. The eigenvalues are the two roots of the equation

$$\begin{vmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 = 0;$$

the field of values $\mathfrak{W}$, i.e. the values of $x_1 \bar{x}_2$ under $x_1 \bar{x}_1 + x_2 \bar{x}_2 = 1$, I determine by setting

$$x_1 = \cos t\,(\cos\phi_1 + i\,\sin\phi_1), \quad x_2 = \sin t\,(\cos\phi_2 + i\,\sin\phi_2);$$

then

$$x_1 \bar{x}_2 = \cos t \sin t[\cos(\phi_1 - \phi_2) + i\,\sin(\phi_1 - \phi_2)] = \frac{1}{2}\sin 2t\, e^{i\,(\phi_1 - \phi_2)}$$

and it assumes, if t, ϕ_1, ϕ_2, vary in every way as real numbers, all values in the disk of radius 1/2 and of center the origin. In the end $M = 1$ since

$$|x_1 y_2| \leq |x_1|\,|y_2| \leq \sqrt{|x_1|^2 + |x_2|^2}\sqrt{|y_1|^2 + |y_2|^2} \leq 1,$$

and, for $x_1 = 1, x_2 = 0$, $y_1 = 0, y_2 = 1$, the value 1 is obtained.

Theorem 7. The example of the bilinear form $x_1 y_2$ shows at the same time:

1. that the field of value $\mathfrak{W}$ for a non-normal bilinear form need not be the smallest convex region which encloses the eigenvalues (because these reduce to the zero point, while $\mathfrak{W}$ is the disk of radius 1/2 with center at the origin);

2. that indeed $W = \frac{1}{2}M$ can happen, so that the constant $\frac{1}{2}$ in Theorem 6 cannot be replaced by a larger one (because $W = \frac{1}{2}, M = 1$)[10].

§4
The boundary of the region $\mathfrak{W}$.

Due to the disproving example from Theorem 7, it is not a self-evident conclusion from Theorem 5 if I now prove for every, even non-normal bilinear form:

Theorem 8. The boundary of the region $\mathfrak{W}$ is a convex curve.

For the proof I represent C according to Theorem 2 in the form $H + i\,K$ where H and K are Hermitian forms, and transform in the known way H by

[10] (Note, during correction.) Mr. I. Schur, to whom the manuscript of the work was submitted, was able to deepen the remarks of this theorem considerably by characterizing all bilinear forms for which $W = M$ and, on the other hand, all for which $W = \frac{1}{2}M$.

a unitary transformation to the form $\mathsf{H} = \mu_1\xi_1\bar{\xi}_1 + \cdots + \mu_n\xi_n\bar{\xi}_n$; $\mu_1, \ldots, \mu_n$ will then be real and the eigenvalues of H and H at the same time; they may be ordered as: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The same unitary transformation transforms K into some other Hermitian form $\mathsf{K}(\xi, \eta)$ and C into a bilinear form

$$\Gamma(\xi, \eta) = \mathsf{H}(\xi, \eta) + i\,\mathsf{K}(\xi, \eta) = \mu_1\xi_1\eta_1 + \cdots + \mu_n\xi_n\eta_n + i\,\mathsf{K}(\xi, \eta)$$

which has the same field of values as $C(x, y)$, μ_1 is the right-most of all values that $\mathsf{H}(\xi, \bar{\xi})$ can take for $\sum \xi_\alpha\bar{\xi}_\alpha = 1$; and since $\mathsf{H}(\xi, \bar{\xi})$ is the real part of $\Gamma(\xi, \bar{\xi})$, $\mathfrak{W}$ lies is on the left of the vertical with the abscissa μ_1, and reaches it eventually. Now if μ_1 is a simple eigenvalue of H, i.e. $\mu_1 > \mu_2 \geq \cdots \geq \mu_n$, then H takes the value μ_1 only if $\xi_1\bar{\xi}_1 = 1$, $\xi_2 = 0, \ldots, \xi_n = 0$; thus, only for these values, $\Gamma(\xi, \eta)$ reaches the mentioned vertical. Now the imaginary part of Γ, i.e. $\mathsf{K}(\xi, \bar{\xi})$, has, for these values, a very specific value. So, if μ_1 is a simple eigenvalue of H, the vertical contains only a single point of $\mathfrak{W}$. However, if μ_1 is a multiple eigenvalue, say $\mu_1 = \cdots = \mu_\alpha > \mu_{\alpha+1} \geq \cdots \geq \mu_n$, then $\Gamma(\xi, \bar{\xi})$ reaches the vertical if $\xi_{\alpha+1} = 0, \ldots, \xi_n = 0$ and $\xi_1\bar{\xi}_1 + \cdots + \xi_\alpha\bar{\xi}_\alpha = 1$. $\mathsf{K}(\xi, \bar{\xi})$ will generally not assume a single value for these values; but it is reduced to a Hermitian form of $\xi_1, \ldots, \xi_\alpha$ alone, which are only constrained by the condition $\xi_1\bar{\xi}_1 + \cdots + \xi_\alpha\bar{\xi}_\alpha = 1$, and the values that $\mathsf{K}(\xi, \bar{\xi})$ assumes, will, because of the known fact mentioned at the beginning of the proof of Theorem 3, be on a connected segment.

The result so far is that the vertical which just touches the region $\mathfrak{W}$ from the right contains either only a single point or a single segment. But if one understands by η any complex number of modulus 1 and if one applies to $\eta C(x, y)$ the same consideration as to $C(x, y)$, then the field of values of this form is compared against that of C by the segment from η to the origin, and it follows that for every other direction at $\mathfrak{W}$ the same applies that has been shown so far for the vertical direction. From this it follows, however, that the outer boundary of the region $\mathfrak{W}$ is a convex curve, which of course can consist also of piecewise straight lines.

§5.
The binary case.
Extension of the task: the region $\mathfrak{B}_n$.

The question of whether $\mathfrak{W}$ fills the whole interior of its outer convex boundary or has holes, has been left open in §4. I now take an extension of the whole question, which shows for the binary the absence of such holes, but at the same time reveals the difficulties which arise in the general case from the possibility of holes.

Instead of two Hermitian forms H, K I consider ν Hermitian forms[11]

[11]In the same direction lies a generalization that Mr. Fejér makes in his work from $\nu = 2$ to $\nu = 3$

$H_1, \dots, H_\nu$ and interpret $u_1 = H_1(x, \bar{x}), \dots, u_\nu = H_\nu(x, \bar{x})$ as real coordinates in the space of ν dimensions; if the x_α vary, now according to the constraint $\sum x_\alpha \bar{x}_\alpha = 1$, then the $u_1, \dots, u_\nu$ span a region whose outer boundary, just like in §4, is convex. For $\nu = 2$ this is exactly what we did above. On the other hand, it will suffice to choose ν equal to the maximum number of linearly independent Hermitian forms that can occur for n variables, namely $\nu = n^2$. All smaller cases one can derive from this one by projection. In this highest case, however, one obtains a region, which may be called $\mathfrak{B}_n$, in the space of n^2 dimensions that is completely determined except for linear homogeneous transformations. So far it is certain that it is finite and convexly bounded from the outside. For the binary domain it now holds:

Theorem 9. The region $\mathfrak{B}_2$, is an ordinary ellipsoid, thus located in a three-dimensional linear subspace of the space of four dimensions.

Let

$$u_1 = x_1\bar{x}_1,\ u_2 = x_2\bar{x}_2,\ u_3 = \frac{1}{2}(x_1\bar{x}_2 + x_2\bar{x}_1),\ u_4 = \frac{1}{2i}(x_1\bar{x}_2 - x_2\bar{x}_1)$$

be chosen as the four linearly independent binary Hermitian forms, then $x_1\bar{x}_1 + x_2\bar{x}_2 = 1$ will be satisfied exactly when setting

$$x_1 = \cos t\, e^{i\,\phi_1}, \quad x_2 = \sin t\, e^{i\,\phi_2}$$

and varying t, ϕ_1, ϕ_2, in every way between 0 and 2π in real terms. Then,

$$u_1 = \cos^2 t,\ u_2 = \sin^2 t,\ u_3 = \frac{1}{2}\sin 2t\cos(\phi_1 - \phi_2),\ u_4 = \frac{1}{2}\sin 2t\sin(\phi_1 - \phi_2).$$

So instead of ϕ_1, ϕ_2, only the difference $\phi_1 - \phi_2$ is essential and $\mathfrak{B}_2$ becomes not three, but two-dimensional. Now, on the other hand,

$$u_3^2 + u_4^2 = u_1u_2, \quad u_1 + u_2 = 1,$$

and it is easy to overlook the fact that the entire surface of this ellipsoid is filled by the above parametric representation. The case $\nu = 2$ is obtained by parallel projection of this ellipsoid; an infinitely distant straight line serves as the projection center, the ∞^2 two-dimensional planes through that infinitely distant straight line as projection rays, and it is projected onto a two-dimensional plane. The projection is therefore either the area of an ellipse or a segment or a point. If one combines this result with a simple calculation, which in the binary case can be linked to the normal form given by Schur's theorem, one has:

Theorem 10. For a binary bilinear form, $\mathfrak{W}$ is the area of an ellipse with the two eigenvalues of the form as focal points, which reduces to the distance between these two eigenvalues if and only if the form is normal.

For the general case $\nu > 2$ or $n > 2$ it has become clear that the area can be hollow. Whether this already occurs for $\nu = 2$, $n > 2$ remains an open question, even if it is unlikely.

§6

The connection with Fejér's theorem.

The way on which I arrived from Fejér's theorem to the statement of Theorem 5 is the following. Introducing the complex variable $z = \cos t + i \sin t$, I combine the pair of Fourier series to a Laurentian expansion $\sum c_n z^n = f(t) + i\,g(t)$; to this I assign the bilinear form of infinitely many variables $\sum_{\beta-\alpha} x_\alpha y_\beta$ (α, β run from $-\infty$ to ∞), a so-called L-form[12], and prove:

1. the spectrum of this L-form, i.e. the totality of its eigenvalues, is the field of values of the function $\sum c_n z^n$ for $|z| = 1$[13].

2. the nth arithmetic mean of the series $\sum c_n z^n$, i.e. the expression

$$c_0 + \frac{n-1}{n}\left(c_1 z + c_{-1}\frac{1}{z}\right) + \cdots + \frac{1}{n}\left(c_{n-1}z^{n-1} + c_{-(n-1)}\frac{1}{z^{n-1}}\right)$$

is a value of the L-form field of values for $|z| = 1$. Because if one sets

$$x_1 = \frac{1}{\sqrt{n}},\ x_2 = \frac{\bar{z}}{\sqrt{n}},\ \cdots,\ x_n = \frac{\bar{z}^{n-1}}{\sqrt{n}},\ x_{n+1} = 0, \ldots,$$

in this, $C(x, \bar{x})$ becomes indeed equal to the just listed mean, when $x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots = 1$[14].

3. Every L-form is normal[15]. For every two L-forms commute and the accompanying form of an L-form is itself an L-form; thus every L-form commute with its accompanying one.

Accordingly, Fejer's theorem is contained in the following: *The field of values $\mathfrak{W}$ of an L-form lies entirely in the smallest convex region $\mathfrak{R}$, which encloses the field of values of the corresponding function $\sum c_n z^n$ along the unit circle $|z| = 1$.*

Conversely, the methods of Fejér's work are sufficient to prove the theorem in this extended version, which I do not go into in detail.

It should also be mentioned that by careful other methods Theorem 4 and Theorem 8 can be extended to arbitrary bounded bilinear forms of infinitely many variables and added that in such a form which is normal, the smallest convex region containing the spectrum belongs certainly to $\mathfrak{W}$ and is not only located within an outer boundary curve.

Kiel, May 10, 1918.

[12] Math. Annalen 70, p. 354.
[13] op. cit.. Theorem 5, p. 360.
[14] op. cit. p. 358, Gött. Nachrichten 1910, §4
[15] op. cit. p. 857.