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003

A REVIEW ON THE INVERSE OF SYMMETRIC TRIDIAGONAL AND BLOCK TRIDIAGONAL MATRICES*

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Dedicated to Gene H. Golub on the occasion of his 60th birthday

Abstract. In this paper some results are reviewed concerning the characteri zation of inverses of symmetric tridiagonal and block tridiagonal matrices as well as results concerning the decay of the elements of the inverses. These results are obtained by relating the elements of inverses to elements of the Cholesky decompositions of these matrices. This gives explicit formulas for the elements of the inverse and gives rise to stable algorithms to compute them. These expressions also lead to bounds for the decay of the elements of the inverse for problems arising from discretization schemes.

Key words. block tridiagonal matrices, decay of elements

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1. Introduction.

When solving elliptic or parabolic partial differential equations (pde's) with finite difference methods, we have to consider tridiagonal (for one-dimensional (1D) problems) or block tridiagonal (for higher dimensions) matrices. For developing and studying preconditioners for iterative methods like the conjugate gradient method, it is often of interest to know the properties of the inverse, for instance, how the elements of the inverse decay along a row or a column; see [13], [14], [17].

Inverses of tridiagonal matrices have been extensively studied in the past, although it seems that most of the results that have been obtained were unrelated and that many of the authors did not know each others' results. To mention just a few, let us cite [1], [2], [5], [6], [18], [19], [24], [26], [29], and [34], where formulas are given for inverses of tridiagonal matrices and [3], [9], [10], [24], [29], [31], and [32], where extensions to block tridiagonal or banded matrices are provided.

Closed form explicit formulas for elements of the inverses can only be given for special matrices, e.g., Toeplitz tridiagonal matrices [19] corresponding, for instance, to constant coefficients 1D partial differential elliptic equations, or for block matrices arising from separable 2D elliptic pde's [3]. We recall that a Toeplitz matrix is a matrix with constant diagonals.

Basically there are two kinds of papers: the first gives analytic formulas for special cases; the second gives characterizations of matrices whose inverse has certain properties, e.g., being tridiagonal or banded.

Historically, the oldest paper we found considering the explicit inverse of matrices is that of Moskovitz [29] from 1944, in which analytic expressions are given for 1D and 2D Poisson model problems. A very important paper for the inverses of band matrices is the seminal 1959 work by Asplund [1] in which conditions under which the inverse of a matrix is banded were given. In the 1960 paper by Bickley and McNamee [10], formulas were given for the 2D problem and separable equations. In 1969, Fischer and Usmani [19] gave a general analytical formula for symmetric Toeplitz tridiagonal matrices, i.e., for 1D model problems. In 1971, Baranger and Duc–Jacquet [5] considered symmetric factorizable matrices (whose elements are $a_i b_j$ for $i \le j$) and

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proved that the inverse is tridiagonal (this is Asplund's result) and conversely. In 1973, Bukhberger and Emel'yanenko [11] gave formulas based on Cholesky factorization for the inverse of a symmetric tridiagonal matrix. Two 1977 papers, by Bank and Rose [3] and Bank [4], gave analytic formulas for the inverse of block tridiagonal matrices arising from separable problems. The 1979 Ikebe paper [24] studied inverses of Hessenberg matrices; specialization of this result to tridiagonal matrices gave Asplund's results. This result is extended to block tridiagonal matrices when the outer blocks are nonsingular. In 1979, Barrett [6] introduced the "triangle property" (a matrix R has this property if $R_{ij} = (R_{ik}R_{kj})/R_{kk}$); a matrix having the "triangle property" and nonzero diagonal elements has a tridiagonal inverse and vice versa). This result is not strictly equivalent to Asplund's result. In one theorem there is a restriction on the diagonal elements and in the other there is a restriction on the nondiagonal elements of the inverse. Also in 1979, Yamamoto and Ikebe [34] obtained formulas for the inverses of banded matrices. Fadeev [18] gave another proof of Ikebe's result for Hessenberg matrices in 1981. Another important paper is that by Barrett and Feinsilver [7]. It established a correspondence between the vanishing of a certain set of minors of a matrix and the vanishing of a related set of minors of the inverse. This gave a characterization of inverses of banded matrices; for tridiagonal matrices this reduces to the "triangle property." In 1984 Barrett and Johnson [8] generalized the work of Barrett and Feinsilver. Also in 1984, Rizvi [32] generalized the "triangle property" to block matrices and gave expressions for inverses of block tridiagonal matrices. The 1987 paper by Rózsa [31] generalized Asplund's work. In 1986, Romani [30] studied the additive structure of the inverses of banded matrices, namely, that the inverse of a $2k+1$ diagonal symmetric banded matrix can be expressed as a sum of k symmetric matrices belonging to the class of inverses of symmetric irreducible tridiagonal matrices. In 1988, Bevilacqua, Codenotti, and Romani [9] gave formulas for block Hessenberg and block tridiagonal matrices with nonsingular outer blocks.

Regarding the decay of the elements of inverses the most interesting papers are those by Demko [15], in which results are proved for particular banded matrices, and by Demko, Moss, and Smith [16], which presents results for positive definite banded matrices. In 1987, Greengard [22] studied the decrease of Green's functions which is equivalent to studying the inverse of the 2D and 3D Poisson problems. Eijkhout and Polman [17] in 1988 exhibited bounds for the inverses of M-matrices, the Cholesky factors of which are bounded by diagonally dominant Toeplitz matrices. These matrices arise in the design of block preconditioners (cf. [13]). Also in 1988, Kuznetsov [25] gave results on the decay of the elements of the inverse for symmetric positive definite matrices that are used in a domain decomposition method (cf. Meurant [28]).

When no explicit solutions for the elements of the inverse can be found, they are usually given in terms of solutions of second-order linear recurrences [5], [9], [14]. However, as it was shown in Concus and Meurant [14] for tridiagonal and pentadiagonal matrices, these recurrences can be numerically unstable and can lead to trouble for large problems. In this paper we obtain most of the previously known results as well as new ones using a unified framework. Simple relationships between elements of the inverse and Cholesky or block Cholesky decompositions are obtained. This allows us to obtain analytic formulas and to compute elements of the inverse in a very stable way, at least when the matrix is symmetric and positive definite. We also provide estimates of decays of the elements of the inverse. It is clear that most of our results can be easily extended to nonsymmetric matrices with straightforward modifications.

The outline of the paper is as follows: in §2, we study the tridiagonal case corresponding to one-dimensional pde problems, in particular, simple but precise formulas for the decay of the elements of the inverse are given. Section 3 is devoted to block tridiagonal matrices. We solve the general problem and as a consequence we easily get formulas for separable two-dimensional elliptic problems. In §4, results about the decay of the inverse are recalled and it is shown how to obtain estimates for two-dimensional pde problems.

Throughout the paper, it is supposed that the matrices under consideration are nonsingular and that their Cholesky decompositions exist. So, the principal minors of the matrices are also nonsingular.

2. Tridiagonal matrices. We are interested in finding formulas for the inverse of a symmetric tridiagonal matrix T of order n,

$$T = \begin{pmatrix} a_1 & -b_2 & & & \\ -b_2 & a_2 & -b_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1} & a_{n-1} & -b_n \\ & & & -b_n & a_n \end{pmatrix}.$$

As a particularly interesting case for pde's, the example of a tridiagonal Toeplitz matrix will be considered:

$$T_a = \begin{pmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{pmatrix}.$$

It should be noted that this latter case has been previously studied by several authors; see, for instance, [19] and [29]. A good reference for numerical methods for solving Toeplitz linear systems is [12].

2.1. The general case. It is natural to suppose that $b_i \neq 0$, for all $i \geq 2$ (that is, T is irreducible) as if one of the b_i's is 0, then the problem can be reduced to two smaller subproblems (for a discussion of this issue, see [6]). Here, the $-$ sign is just a technical convenience and has no specific significance, unless otherwise stated. From [1], [5], and [18] it is known that there exist two sequences $\{u_i\}, \{v_i\}, i = 1, n$ such that

$$T^{-1} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \dots & u_1v_n \\ u_1v_2 & u_2v_2 & u_2v_3 & \dots & u_2v_n \\ u_1v_3 & u_2v_3 & u_3v_3 & \dots & u_3v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1v_n & u_2v_n & u_3v_n & \dots & u_nv_n \end{pmatrix}.$$

This result can also be easily proved with the techniques used in §3. Moreover, every nonsingular matrix of the previous form (the matrices of this class have been called "matrices factorisables" in [5]) is the inverse of an irreducible tridiagonal matrix. It means that to know all the elements of T^{-1}, it is enough to compute its first and last columns. In fact, it is enough to know $2n-1$ quantities as u_1 can be chosen arbitrarily (note that $2n-1$ is the number of nonzero terms determining T). The second-order recurrences for computing u_i and v_i given in [13] can be unstable and can lead to trouble for large systems, but this problem was already solved in [14]. However, much simpler formulas can be obtained if $\{u_i\}, \{v_i\}$ are computed in the following way: let us first compute v. The $\{u_i\}, \{v_i\}$ are only defined up to a multiplicative constant. So, for instance, u_1 can be chosen as $u_1 = 1$. Then let $v = (v_1, \cdots, v_n)^T$; as $u_1 = 1$ the first column of T^{-1} is v, so

$$Tv = e_1$$

where $e_1 = (1, 0, \cdots, 0)^T$.

Because of the special structure of the right-hand side, it is natural to consider a UL decomposition of T:

$$T = U D_U^{-1} U^T,$$

with

$$U = \begin{pmatrix} d_1 & -b_2 & & & \\ & d_2 & -b_3 & & \\ & & \ddots & \ddots & \\ & & & d_{n-1} & -b_n \\ & & & & d_n \end{pmatrix}, \qquad D_U = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{pmatrix}.$$

By inspection, it is easily seen that

$$d_n = a_n, \quad d_i = a_i - \frac{b_{i+1}^2}{d_{i+1}}, \quad i = n-1, \cdots, 1.$$

With the help of the UL decomposition the linear system for v can be easily solved.

PROPOSITION 2.1.

$$v_1 = \frac{1}{d_1}, \quad v_i = \frac{b_2 \cdots b_i}{d_1 \cdots d_{i-1} d_i}, \quad i = 2, \cdots, n.$$

Proof. It is clear that solving $Tv = e_1$ is equivalent to solving

$$D_U^{-1} U^T v = \frac{1}{d_1} e_1,$$

and the proposition follows. $\square$

Let $u = (u_1, \cdots, u_n)^T$; the last column of T^{-1} is $v_n u$ and therefore

$$v_n T u = e_n,$$

where $e_n = (0, \cdots, 0, 1)^T$. To solve this system, because of the structure of the right-hand side, it is easier to use an LU decomposition of T:

$$T = L D_L^{-1} L^T,$$

with

$$L = \begin{pmatrix} \delta_1 & & & & \\ -b_2 & \delta_2 & & & \\ & \ddots & \ddots & & \\ & & -b_{n-1} & \delta_{n-1} & \\ & & & -b_n & \delta_n \end{pmatrix}, \qquad D_L = \begin{pmatrix} \delta_1 & & & & \\ & \delta_2 & & & \\ & & \ddots & & \\ & & & \delta_{n-1} & \\ & & & & \delta_n \end{pmatrix}.$$

By inspection,

$$\delta_1 = a_1, \quad \delta_i = a_i - \frac{b_i^2}{\delta_{i-1}}, \quad i = 2, \cdots, n.$$

PROPOSITION 2.2.

$$u_n = \frac{1}{\delta_n v_n}, \quad u_{n-i} = \frac{b_{n-i+1} \cdots b_n}{\delta_{n-i} \cdots \delta_n v_n}, \quad i = 1, \cdots, n-1.$$

Proof. Clearly,

$$D_L L^T u = \frac{1}{\delta_n v_n} e_n.$$

Solving for u gives the result. □

Note that

$$u_1 = \frac{b_2 \cdots b_n}{\delta_1 \cdots \delta_n v_n} = \frac{d_1 \cdots d_n}{\delta_1 \cdots \delta_n},$$

but $d_1 \cdots d_n = \delta_1 \cdots \delta_n = \det T$, so $u_1 = 1$, as the values of $\{v_i\}$ were computed with this scaling.

Together, the preceding results prove the following theorem.

THEOREM 2.3. *For the general case,*

$$(T^{-1})_{i,j} = u_i v_j = b_{i+1} \cdots b_j \frac{d_{j+1} \cdots d_n}{\delta_i \cdots \delta_n} \quad \forall i, \quad \forall j > i,$$
$$(T^{-1})_{i,i} = u_i v_i = \frac{d_{i+1} \cdots d_n}{\delta_i \cdots \delta_n} \quad \forall i.$$

In these products, terms that have indices greater than n must be taken equal to 1.

This gives a computationally stable and simple algorithm for computing elements of the inverse of T as it involves only Cholesky decompositions that are proved to be stable when the matrix T possesses enough properties as to be diagonally dominant.

We are also interested in characterizing the decrease of the elements of T^{-1} along a row or column starting from the diagonal element. In [13], it is proved that if T is strictly diagonally dominant, then the sequence $\{u_i\}$ is strictly increasing and the sequence $\{v_i\}$ is strictly decreasing. From Theorem 2.3, we have

$$\frac{(T^{-1})_{i,j}}{(T^{-1})_{i,j+1}} = \frac{d_{j+1}}{b_{j+1}},$$

and, therefore,

$$(T^{-1})_{i,j} = \frac{d_{j+1} \cdots d_{j+l}}{b_{j+1} \cdots b_{j+l}} T^{-1}_{i,j+l}.$$

By induction, the following result is proved.

THEOREM 2.4. *If T is strictly diagonally dominant ($a_i > b_i + b_{i+1}$, for all i) then the sequence d_i is such that*

$$d_i > b_i.$$

Hence, the sequence $T^{-1}_{i,j}$ is a strictly decreasing function of j, for $j > i$. Similarly, we have $\delta_i > b_{i+1}$.

Proof. We have

$$d_n = a_n > b_n.$$

Suppose $d_{i+1} > b_{i+1}$; then

$$d_i = a_i - \frac{b_{i+1}^2}{d_{i+1}} > a_i - b_{i+1} > b_i. \qquad \Box$$

Remark that the theorem can be proved under a weaker hypothesis. Namely, we can only suppose that T is diagonally dominant ($a_i \geq b_i + b_{i+1}$) and $a_n > b_n$, $a_1 > b_2$. This result was already proven in [13], although not in the same way.

2.2. The Toeplitz case. Here, as an example, the Toeplitz tridiagonal matrix T_a that was defined in the introduction of §2 is considered. The interesting thing is that we are then able to analytically solve the recurrences arising in the Cholesky decompositions. This is given in the following lemma.

LEMMA 2.5. *Let*

$$\alpha_1 = a, \quad \alpha_i = a - \frac{1}{\alpha_{i-1}}, \quad i = 2, \cdots, n.$$

Then, if $a \neq 2$,

$$\alpha_i = \frac{r_+^{i+1} - r_-^{i+1}}{r_+^i - r_-^i},$$

where

$$r_\pm = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

are the two solutions of the quadratic equation $r^2 - ar + 1 = 0$. *If* $a = 2$, *then* $\alpha_i = (i+1)/i$.

Proof. We set

$$\alpha_i = \frac{\beta_i}{\beta_{i-1}}.$$

Therefore, we now have a recurrence on β_i:

$$\beta_i - a\beta_{i-1} + \beta_{i-2} = 0, \quad \beta_0 = 1, \quad \beta_1 = a.$$

The solution of this linear second-order difference equation is well known (see, for instance, [23]):

$$\beta_i = c_0 r_+^{i+1} + c_1 r_-^{i+1}.$$

From the initial conditions we have $c_0 + c_1 = 0$. Hence, the solution can be written as

$$\beta_i = c_0(r_+^{i+1} - r_-^{i+1});$$

when $a = 2$, it is easy to see that $\beta_i = i + 1$, and the result follows. $\Box$

This proof explains the difficulties that arise when using the second-order recurrences for u_i and v_i. As $r_+ > 1$, $\beta_i \to \infty$ when $i \to \infty$. On the contrary, α_i remains bounded.

Remark. α_i can be written in the other form,

$$\alpha_i = \frac{\sinh((i+1)\psi)}{\sinh(i\psi)}, \quad \text{where } \cosh(\psi) = \frac{a}{2} \quad \text{if } a > 2,$$

$$\alpha_i = \frac{\sin((i+1)\psi)}{\sin(i\psi)}, \quad \text{where } \cos(\psi) = \frac{a}{2} \quad \text{if } a < 2.$$

From this lemma, the solutions of the recurrences involved in the Cholesky decompositions of T_a can be deduced. When $a \neq 2$, we have

$$d_{n-i+1} = \frac{r_+^{i+1} - r_-^{i+1}}{r_+^i - r_-^i}.$$

Solving for v the following result is obtained.

PROPOSITION 2.6. *For the sequence v_i in T_a^{-1},*

$$v_i = \frac{r_+^{n-i+1} - r_-^{n-i+1}}{r_+^{n+1} - r_-^{n+1}} \quad \forall i.$$

Note in particular, that

$$v_n = \frac{r_+ - r_-}{r_+^{n+1} - r_-^{n+1}}.$$

It is obvious that for the Toeplitz case, we have the relation

$$\delta_i = d_{n-i+1}.$$

Solving for u, the following result is obtained.

PROPOSITION 2.7. *For the sequence u_i in T_a^{-1},*

$$u_i = \frac{r_+^i - r_-^i}{r_+ - r_-}, \qquad i = 1, \cdots, n.$$

With these two last results, the expression of the elements of the inverse can be computed.

THEOREM 2.8. *For $j \geq i$ and when $a \neq 2$,*

$$(T_a{}^{-1})_{i,j} = u_i v_j = \frac{(r_+^i - r_-^i)(r_+^{n-j+1} - r_-^{n-j+1})}{(r_+ - r_-)(r_+^{n+1} - r_-^{n+1})},$$

where $r_\pm$ are the two solutions of the quadratic equation $r^2 - ar + 1 = 0$. This can also be written (for $a > 2$) as

$$(T_a{}^{-1})_{i,j} = \frac{\sinh(i\psi)\ \sinh((n-j+1)\psi)}{\sinh(\psi)\ \sinh((n+1)\psi)}, \quad \textit{with } \cosh(\psi) = \frac{a}{2};$$

for $a = 2$, we have

$$(T_a)_{i,j}^{-1} = i\ \frac{n-j+1}{n+1}.$$

These formulas are similar to the ones in [19], where they were obtained with a different method.

Regarding the decay of the elements of T_a^{-1}, in this simple case we can obtain useful bounds. Suppose that $a > 2$. Then we have

$$\frac{u_i v_j}{u_i v_{j+1}} = \frac{r_+^{n-j+1} - r_-^{n-j+1}}{r_+^{n-j} - r_-^{n-j}} = \frac{r_+^{n-j+1}}{r_+^{n-j}} \left(\frac{1 - r^{n-j+1}}{1 - r^{n-j}} \right) > r_+ > 1$$

and

$$u_i v_j < r_+^{i-j-1} \frac{(1-r^i)(1-r^{n-j+1})}{(1-r)(1-r^{n+1})}, \qquad j \geq i+1,$$

where $r = (r_-/r_+) < 1$.

From this, the following result can be deduced.

THEOREM 2.9. *If $a > 2$, we have the bound*

$$({T_a}^{-1})_{i,j} < (r_-)^{j-i} ({T_a}^{-1})_{i,i} \quad \forall\, i, \quad \forall j \geq i,$$

$$({T_a}^{-1})_{i,j} < \frac{r_-^{j-i+1}}{1-r} \quad \forall\, i, \quad \forall j \geq i+1.$$

This implies that the following estimate holds: let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be given:

$$\frac{({T_a}^{-1})_{i,j}}{({T_a}^{-1})_{i,i}} \leq \epsilon_1 \quad \text{if } j-i \geq \frac{\log \epsilon_1^{-1}}{\log r_+},$$

$$({T_a}^{-1})_{i,j} \leq \epsilon_2 \quad \text{if } j-i+1 \geq \frac{\log\left[\epsilon_2(1-r)\right]^{-1}}{\log r_+}.$$

3. Block tridiagonal matrices. In this section we consider the symmetric block tridiagonal matrix

$$A = \begin{pmatrix} D_1 & -A_2^T & & & \\ -A_2 & D_2 & -A_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & -A_{n-1} & D_{n-1} & -A_n^T \\ & & & -A_n & D_n \end{pmatrix}.$$

Each block is of order n, although this is not essential for our results.

In the two-dimensional partial differential applications we have in mind, the matrices D_i will be tridiagonal and the matrices A_i will be diagonal, but this does not influence the method and the results that will be described in this section.

As an interesting example, the following problem will be considered:

$$A_T = \begin{pmatrix} T & -I & & & \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ & & & -I & T \end{pmatrix},$$

T being a Toeplitz tridiagonal matrix. This example arises, for instance, from the discretization of the Poisson equation in a square.

3.1. The general case. To obtain the formulas for the inverse, three different block factorizations will be used: LU, UL, and a twisted factorization. Let us first give formulas for the block LU and UL factorizations. Denote by L the block lower part of A. Then,

$$A = (\Delta + L)\ \Delta^{-1}\ (\Delta + L^T) = (\Sigma + L^T)\ \Sigma^{-1}\ (\Sigma + L),$$

where Δ and Σ are block diagonal matrices whose diagonal blocks are denoted by Δ_i and Σ_i and are given by block recurrences

$$\begin{cases} \Delta_1 = D_1, \\ \Delta_i = D_i - A_i \, (\Delta_{i-1})^{-1} \, (A_i)^T, \end{cases} \qquad \begin{cases} \Sigma_n = D_n, \\ \Sigma_i = D_i - (A_{i+1})^T \, (\Sigma_{i+1})^{-1} \, A_{i+1}. \end{cases}$$

A twisted factorization can be defined for each $j = 2, \cdots, n-1$ as

$$A = (\Phi + \mathcal{L})\Phi^{-1}(\Phi + \mathcal{L}^T)$$

where Φ is a block diagonal matrix and $\mathcal{L}$ has the following twisted block structure:

$$\mathcal{L} = \begin{pmatrix} 0 & & & & & & \\ -A_2 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & -A_j & 0 & -A_{j+1}^T & & \\ & & & & \ddots & \ddots & \\ & & & & & 0 & -A_n^T \\ & & & & & & 0 \end{pmatrix},$$

where the row with two nonzero terms is the jth block row. By inspection, we have

$$\begin{aligned} \Phi_i &= \Delta_i, \qquad i = 1, \cdots, j-1, \\ \Phi_i &= \Sigma_i, \qquad i = n, \cdots, j+1, \\ \Phi_j &= D_j - A_j \Delta_{j-1}^{-1} A_j^T - A_{j+1}^T \Sigma_{j+1}^{-1} A_{j+1}. \end{aligned}$$

It should be noted that when we know the LU and UL decompositions, we know the twisted factorizations for all j's. So, these twisted factorizations are only a convenience to obtain simpler formulas.

With the twisted factorization at hand, the block jth column X of the inverse can be computed in a straightforward way.

THEOREM 3.1. *The jth block column X of A^{-1} is given by*

$$\begin{aligned} X_j &= \Phi_j^{-1}, \\ X_{j-l} &= \Delta_{j-l}^{-1} \, A_{j-l+1}^T \, \Delta_{j-l+1}^{-1} \cdots \Delta_{j-1}^{-1} \, A_j^T \, \Phi_j^{-1}, \qquad l = 1, \cdots, j-1, \\ X_{j+l} &= \Sigma_{j+l}^{-1} \, A_{j+l} \, \Sigma_{j+l-1}^{-1} \cdots \Sigma_{j+1}^{-1} \, A_{j+1} \, \Phi_j^{-1}, \qquad l = 1, \cdots, n-j. \end{aligned}$$

These expressions are valid for any block tridiagonal matrix that satisfies our hypothesis. When matrices A_i are nonsingular, A is said to be *proper* (cf. [9]); in this case, the formulas can be simplified. Using the uniqueness of the inverse, we can prove the following.

PROPOSITION 3.2. *If A is proper, then*

$$\begin{aligned} \Phi_j^{-1} &= A_{j+1}^{-1} \, \Sigma_{j+1} \cdots A_n^{-1} \, \Sigma_n \, \Delta_n^{-1} \, A_n \cdots \Delta_{j+1}^{-1} \, A_{j+1} \, \Delta_j^{-1} \\ &= A_j^{-T} \, \Delta_{j-1} \cdots A_2^{-T} \, \Delta_1 \, \Sigma_1^{-1} \, A_2^T \cdots \Sigma_{j-1}^{-1} \, A_j^T \, \Sigma_j^{-1}. \end{aligned}$$

From these relations, we deduce alternate formulas for the other elements of the inverse.

THEOREM 3.3. *If A is proper,*

$$X_{j-l} = (A_{j-l}^{-T}\Delta_{j-l-1}\cdots A_2^{-T}\Delta_1)(\Sigma_1^{-1}A_2^T\cdots A_j^T\Sigma_j^{-1}), \qquad l = 1,\cdots,j-1,$$
$$X_{j+l} = (A_{j+l+1}^{-1}\Sigma_{j+l+1}\cdots A_n^{-1}\Sigma_n)(\Delta_n^{-1}A_n\cdots\Delta_{j+1}^{-1}A_{j+1}\Delta_j^{-1}), \qquad l = 1,\cdots,n-j.$$

As before, the elements of the inverse can be computed in a stable way using block Cholesky decomposition when the matrix is diagonally dominant or positive definite. These formulas are the block counterpart of the ones for tridiagonal matrices in Theorem 2.3. They give a characterization of the inverse of a proper block tridiagonal matrix.

THEOREM 3.4. *If A is proper, there exist two (nonunique) sequences of matrices $\{U_i\}, \{V_i\}$ such that for $j \geq i$*

$$(A^{-1})_{ij} = U_iV_j^T,$$

with $U_i = A_i^{-T}\Delta_{i-1}\cdots A_2^{-T}\Delta_1$ and $V_j^T = \Sigma_1^{-1}A_2^T\cdots A_j^T\Sigma_j^{-1}$.

In other words, A^{-1} can be written as

$$A^{-1} = \begin{pmatrix} U_1V_1^T & U_1V_2^T & U_1V_3^T & \cdots & U_1V_n^T \\ V_2U_1^T & U_2V_2^T & U_2V_3^T & \cdots & U_2V_n^T \\ V_3U_1^T & V_3U_2^T & U_3V_3^T & \cdots & U_3V_n^T \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_nU_1^T & V_nU_2^T & V_nU_3^T & \cdots & U_nV_n^T \end{pmatrix}.$$

This result was proven using different methods in [9].

If we denote by E_n the matrix

$$E_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

and $L_n = E_n^T - I$, $U = (U_1,\cdots,U_n)^T$, $V^T = (V_1^T,\cdots,V_n^T)$, we can write the result of Theorem 3.4 as

$$A^{-1} = UV^T \circ E_n + VU^T \circ L_n,$$

where $\circ$ denotes the Hadamard (element by element) product. If we denote by u_i the columns of U and by v_i those of V, this can also be written as

$$A^{-1} = \sum_{j=1}^{n} \left(u_iv_i^T \circ E_n + v_iu_i^T \circ L_n\right).$$

This result about the additive structure of the inverse was proved for banded matrices in [30].

3.2. Separable problems. Here we consider the matrix A_T defined at the beginning of this section. This is not the most general problem that can be considered, but for this case explicit formulas can be given. Because the matrix is persymmetric, we have

$$\Delta_i = \Sigma_{n-i+1}, \qquad i = 1,\cdots,n,$$

and

$$X_{j-l} = (\Delta_{j-l+1} \cdots \Delta_1)(\Delta_n^{-1} \cdots \Delta_{n+j+1}^{-1}), \qquad l = 1, \cdots, j-1.$$

With the same methods as in [27], it can be proved that all the Δ_i's have the same eigenvectors as T and hence they commute. In this case, the block recursion for the diagonal elements in the Cholesky decomposition can be solved:

$$\Delta_1 = T,$$
$$\Delta_i = T - (\Delta_{i-1})^{-1}.$$

Let $\Lambda = (\lambda^j)$ be the diagonal matrix of the eigenvalues of T and Λ_i the corresponding one for Δ_i. Then the following propositions hold.

PROPOSITION 3.5. *The following relation holds*:

$$\Lambda_1 = \Lambda,$$
$$\Lambda_i = \Lambda - (\Lambda_{i-1})^{-1},$$

which can be written elementwise as

$$\lambda_1^j = \lambda^j,$$
$$\lambda_i^j = \lambda^j - \frac{1}{\lambda_{i-1}^j}.$$

From this last result and Lemma 2.5, we can compute the values of λ_i^j.

PROPOSITION 3.6.

$$\lambda_i^j = \frac{(r(j)_+)^{i+1} - (r(j)_-)^{i+1}}{(r(j)_+)^i - (r(j)_-)^i},$$

where $r(j)_\pm$ are the roots of $r^2 + \lambda^j r - 1 = 0$. If $\lambda^j > 2$ (which is the case for the Poisson problem), this can be written

$$\lambda_i^j = \frac{\sinh((i+1)\psi_j)}{\sinh(i\psi_j)}, \qquad \cosh(\psi_j) = \frac{\lambda^j}{2}.$$

Now let Λ_+ and Λ_- be the diagonal matrices whose diagonal elements are $r(j)_+$ and $r(j)_-$. From Proposition 3.6, we have

$$\Lambda_i = (\Lambda_+^{i+1} - \Lambda_-^{i+1})(\Lambda_+^i - \Lambda_-^i)^{-1}.$$

Let Q be the matrix of the eigenvectors of all the matrices. Denote

$$T_+ = Q\Lambda_+ Q^T, \qquad T_- = Q\Lambda_- Q^T;$$

then

$$\Delta_i = ({T_+}^{i+1} - {T_-}^{i+1})(T_+^i - T_-^i)^{-1}.$$

Along the same lines as what was done for a tridiagonal matrix, we have the following theorem.

THEOREM 3.7. *The (block) elements of the inverse are given by*

$$(A_T^{-1})_{i,j} = ({T_+}^{i+1} - {T_-}^{i+1})({T_+}^{n-j+1} - {T_-}^{n-j+1})\left({T_+}^{n+1} - {T_-}^{n+1}\right)^{-1}(T_+ - T_-)^{-1}.$$

From these results, it can easily be seen that

$$(A_T^{-1})_{i,j} = S_n^{-1}(T)\ S_{i-1}(T)\ S_{n-j}(T) \quad \text{for } j \geq i,$$

where $S_n(x)$ is the shifted Chebyshev polynomial of the second kind, that is, defined for $x > 2$ as

$$S_i(x) = \frac{\sinh((i+1)\psi)}{\sinh(\psi)} \quad \text{with } \cosh(\psi) = x/2.$$

This expression for the inverse was given in [3]. Now we establish relations that will be useful in the next section. The (simple) roots of the Chebyshev polynomial S_n are $\mu_l = 2\cos(l\pi/(n+1)), \quad l = 1, \cdots, n$. Therefore,

$$S_n(x) = \prod_{l=1}^{n}(x - \mu_l).$$

As in [20], remark that for $j \geq i$, $S_n^{-1}(x)\ S_{i-1}(x)\ S_{n-j}(x)$ is a rational function in x, so it can be developed in elementary fractions. We write

$$S_n^{-1}(x)\ S_{i-1}(x)\ S_{n-j}(x) = \sum_{l=1}^{n} \frac{\alpha_{ij}^l}{x - \mu_l}.$$

It can easily be seen that

$$\alpha_{ij}^l = \frac{S_{i-1}(\mu_l)\ S_{n-j}(\mu_l)}{S_n'(\mu_l)}.$$

From this expansion, we get an expression for the elements of the inverse in terms of the zeros of Chebyshev polynomials.

THEOREM 3.8. *The (block) elements of the inverse of A_T are given by:*

$$(A_T^{-1})_{i,j} = \sum_{l=1}^{n} \alpha_{ij}^l\ (T - \mu_l I)^{-1}, \qquad j \geq i.$$

These results can be extended to more general separable problems, although in these cases the roots of the involved polynomials are not explicitly known.

4. Decay of the inverse for two-dimensional problems. In this section, the decay of the elements for two-dimensional pde problems is examined. We recall some known results and establish new ones. First, the Poisson equation is considered that is easy to handle, as the inverse is explicitly known. Then we will turn to the problem of finding bounds for general tridiagonal problems using results from convergence of iterative methods. Finally, the possibility to numerically compute approximate decays for certain block tridiagonal matrices is considered.

4.1. The Poisson equation. Because the Poisson equation is an isotropic problem, it is enough to look at the decay of the elements in one direction of the underlying mesh, i.e., we can only look at the diagonal blocks of the inverse. This is because a

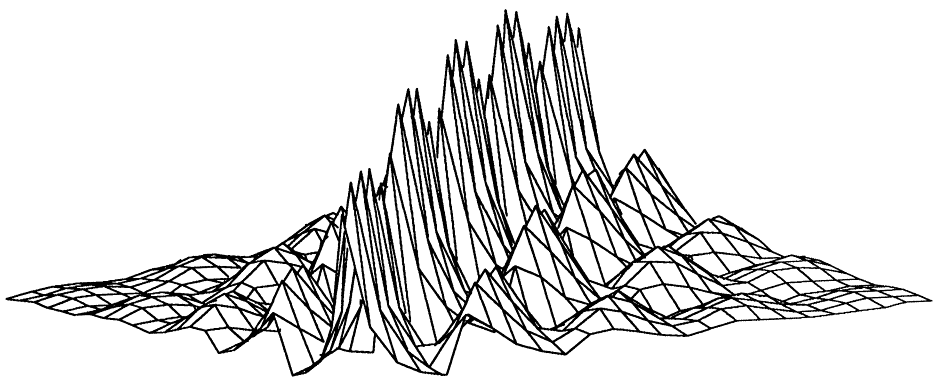

FIG. 1. *Exact inverse of the Poisson problem matrix.*

column (or a row) of the inverse is obtained by putting a Dirac delta function (a "function" being 1 in one point of the mesh and 0 elsewhere) as the right-hand side. Because of the isotropic property of the diffusion equation (as the limit of a time-dependent problem), the right-hand side diffuses the same in both directions. A picture of the inverse of the Poisson problem matrix for a 5×5 mesh is given in Fig. 1.

If we look more closely at what happens for a row of the matrix, starting from the diagonal, we obtain what is shown in Fig. 2. This picture has been obtained for row number 61 in a 121×121 matrix corresponding to a 11×11 mesh.

From the previous section, it is known that

$$(A_T^{-1})_{i,i} = \sum_{l=1}^{n} \alpha_{ii}^l \; (T - \mu_l I)^{-1},$$

where

$$\alpha_{ii}^l = \frac{S_{i-1}(\mu_l)\; S_{n-i}(\mu_l)}{S_n'(\mu_l)}, \qquad \mu_l = 2\cos\left(\frac{l\pi}{n+1}\right) < 2,$$

and

$$T = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

Therefore, $T - \mu_l I$ is a Toeplitz matrix with a diagonal element greater than 2. From §2, we know that the elements of the inverses of all these matrices strictly decay away from the diagonal along a row.

THEOREM 4.1. *Let B be the ith diagonal block of the inverse of A_T and $r_+[l]$ the positive root of $r^2 - (4 - \mu_l)r - 1 = 0$; then*

$$\frac{B_{pp}}{B_{pq}} \geq \min_l \{(r_+[l])^{q-p}\}, \quad q > p.$$

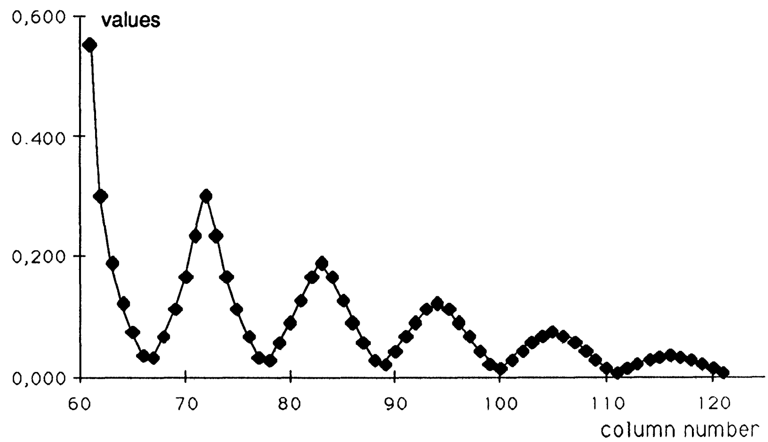

FIG. 2. *A row of the inverse of the Poisson problem matrix.*

Proof. Let $T_l = (T - \mu_l)^{-1}$,

$$B_{pp} = \sum_{l=1}^{n} \alpha_{ii}^{l} (T_l)_{pp} > \sum_{l=1}^{n} \alpha_{ii}^{l} (r_+[l])^{|q-p|} (T_l)_{pq},$$

hence

$$\frac{B_{pp}}{B_{pq}} \geq \frac{\sum_{l=1}^{n} \alpha_{ii}^{l} (r_+[l])^{|q-p|} (T_l)_{pq}}{\sum_{l=1}^{n} \alpha_{ii}^{l} (T_l)_{pq}} \geq \min_{l}\{(r_+[l])^{|q-p|}\}. \qquad \square$$

This gives a uniform (related to i) estimate of the decay.

Asymptotically, we obtain

$$\frac{B_{pq}}{B_{pp}} \leq C(h),$$

where

$$C(h) = 1 - (q-p)\pi h + O(h^2).$$

However, the bound of Theorem 4.1 is a little pessimistic, as shown by the following numerical example. Consider the linear system from the Poisson equation for $n = 11$. Figure 3 shows the relative decrease of the elements for a row of a diagonal block and the bound given by the previous formulas. We see that the slope is correct, but the values are pessimistic.

4.2. The general block tridiagonal case. Here we consider finding bounds for the decay of the elements of the inverse of a general symmetric positive definite tridiagonal matrix corresponding to the discretization of an elliptic or parabolic problem with a five-point finite difference scheme. The matrix of the problem is

$$A = \begin{pmatrix} D_1 & -A_2^T & & & \\ -A_2 & D_2 & -A_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & -A_{n-1} & D_{n-1} & -A_n^T \\ & & & -A_n & D_n \end{pmatrix}.$$

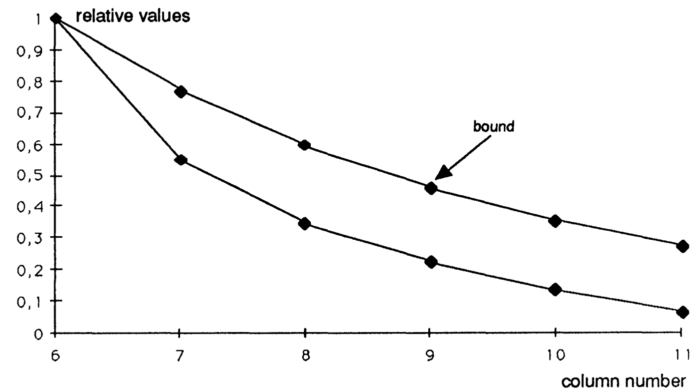

FIG. 3. *Bound for the relative decay for the Poisson problem matrix.*

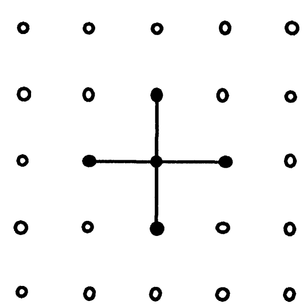

FIG. 4. *Five-point finite difference scheme.*

In this example, matrices D_i are tridiagonal and matrices A_i are diagonal corresponding to the scheme displayed in Fig. 4.

To obtain bounds on the decay of the elements of the inverse, we will follow the same lines as [15]; see also [16]. Consider solving the linear system

$$Ax = b$$

with a Chebyshev first-order iterative method: let x^0 be given and

$$x^{k+1} = x^k + \alpha_k(b - Ax^k).$$

This method converges when A is symmetric positive definite and the coefficients α_k are chosen as the reciprocals of the roots of Chebyshev polynomials.

If $e^k = x - x^k$ is the error we have the following bounds (cf., for instance, [21]).

PROPOSITION 4.2. *Let $\kappa = \lambda_{\max}/\lambda_{\min}$ be the condition number of A; then*

$$\|e^k\|_\infty = \max_i |e_i^k| \le \|e^k\|_2 \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|e^0\|_2.$$

In practice this method is not used because it is unstable; to be stabilized, special orderings of the coefficients must be used. However, we will only use it to obtain bounds on the solution. For this purpose, the previous results can be used to get bounds on the infinity norm. For the Chebyshev method,

$$e^k = P_k(A)e^0,$$

where P_k is a kth-order polynomial. From this, it follows that

$$\|e^k\|_\infty \le \|P_k(A)\|_\infty \|e^0\|_\infty \le \sqrt{N}\|P_k(A)\|_2\|e^0\|_\infty,$$

where $N = n^2$.

Therefore, we have also the following proposition.

PROPOSITION 4.3.

$$\|e^k\|_\infty \le 2n\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \|e^0\|_\infty.$$

To compute the jth column x (or the jth row as the matrix is symmetric) of the inverse, a system with $b = e_j$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)^T$, must be solved, the nonzero element being in position j. Now, consider the Chebyshev iterative method with $x^0 = 0$, so $e^0 = x$.

As $x^0 = 0$, the vector $x^1 = \alpha_1 e_j$ has the same sparsity pattern as e_j. The idea is to consider the sparsity patterns of the successive iterates x^k. To do this, it is easier to think in terms of the underlying two-dimensional mesh. Let the mesh points be indexed by two integers (p, q), and $N(p, q)$ denote the set of neighbouring mesh points in the five-point stencil centered on (p, q). Then, the following proposition holds.

PROPOSITION 4.4. *Let $\mathcal{S}(x^k)$ be the set of mesh points corresponding to the sparsity pattern of x^k. If $\mathcal{S}(e_j) = \mathcal{S}(x^1) = (p_1, q_1)$, then*

$$\mathcal{S}(x^k) = \mathcal{S}(x^{k-1}) \bigcup_{(p_l, q_l) \in \mathcal{S}(x^{k-1})} N(p_l, q_l).$$

Proof. It is clear that the sparsity pattern of x^k is deduced from the sparsity pattern of x^{k-1} by a multiplication with A. The vector x^{k-1} can be written as $x^{k-1} = \sum_l \beta_l e_l$, where the index runs across the sparsity pattern of x_{k-1}. So, the sparsity pattern we are looking for is the union of the sparsity patterns of Ae_l for all l in the sparsity pattern of x^{k-1}. But the vector Ae_l is the lth column of A; therefore, there are at most only five nonzero terms corresponding to the mesh point related to the lth component and its four neighbours in the five-point stencil. □

Remark. This result is not restricted to the five-point stencil and can be easily extended, for instance, to sparse matrices arising from finite element methods.

The result from Proposition 4.4 is illustrated in Fig. 5, the black points corresponding to the nonzero components in x^k.

Let S_j^k be the sets of indices $S(x^k)$ generated from $b = e_j$. Regarding the decay of the elements of the inverse, the following general result holds.

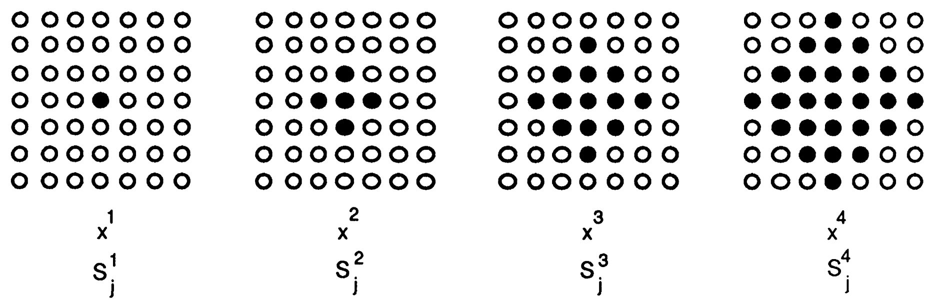

FIG. 5. *Sparsity patterns of x^1, x^2, x^3, and x^4.*

THEOREM 4.5.

$$|A^{-1}_{ij}| \le 2n\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \max_j |A^{-1}_{ij}| \quad \forall i \notin S^k_j.$$

Proof. We have $e^k = x - x^k$; when $i \notin S^k_j$, the corresponding components of x^k are zero. Hence $|A^{-1}_{ij}| \le \|e^k\|_\infty$. □

The condition $i \notin S^k_j$ is verified, for instance, for mesh points (p_i, q_i) satisfying

$$|(p_i, q_i) - (p_j, q_j)| \ge kh,$$

where h is the mesh size. The last theorem shows that the elements of the inverse decay as shown in Fig. 1.

Now we specialize to problems arising from finite difference approximations. We suppose that A is a diagonally dominant M-matrix. Then we have the following result.

PROPOSITION 4.6. *When A is a diagonally dominant M-matrix,*

$$(A^{-1})_{ii} \ge \frac{1}{A_{ii}} \quad \forall i,$$
$$\max_j (A^{-1})_{ij} = (A^{-1})_{ii} \quad \forall i.$$

Proof. Denote $C = A^{-1}$. Looking at the AC product, we obtain

$$1 = \sum_{k=1}^{n} a_{ik}c_{ki} = a_{ii}c_{ii} - \sum_{k\ne i} |a_{ik}|c_{ki},$$

because the a_{ik}, $k \ne i$ are nonpositive and the c_{ki} are positive. Therefore,

$$a_{ii}c_{ii} \ge 1,$$

which gives the first result. Now, suppose there exists a $j \ne i$ such that $\max_k c_{ik} = c_{ij} > c_{ii}$; then

$$0 = \sum_{k=1}^{n} a_{jk}c_{ki} = a_{jj}c_{ij} - \sum_{k\ne j} |a_{jk}|c_{ki}.$$

By hypothesis, $0 \leq c_{ki} = c_{ik} < c_{ij}$, so

$$0 > c_{ij} \left(a_{jj} - \sum_{k \neq j} |a_{jk}| \right).$$

But as $c_{ij} \geq 0$ and A is diagonally dominant, this is a contradiction. Therefore the maximum of the elements of the inverse occurs on the diagonal. □

For the Poisson problem we considered in the last section, κ is explicitly known,

$$\kappa \simeq \frac{1}{\sin(\frac{\pi h}{2})^2}.$$

From Theorem 4.5, we know that

$$\frac{c_{ij}}{c_{ii}} \leq C(h), \quad \text{where } C(h) = \frac{2}{h}(1 - k\pi h).$$

This is a factor of $2/h$ off from the formula we obtained in §4.1, which can be quite large. However, this general formula must account for the possible worst case for the decay.

For a general diagonally dominant M-matrix, we have the following theorem.

THEOREM 4.7. *When A is a diagonally dominant M-matrix,*

$$\frac{(A^{-1})_{ij}}{(A^{-1})_{jj}} \leq 2n \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k \qquad \forall i \notin S_j^k.$$

Still, this bound is not very satisfactory when it is compared, for instance, with the bound obtained for the Poisson problem. In order to obtain more insights into particular problems, we will specialize to generalized strictly diagonally dominant matrices.

DEFINITION 4.8. A is generalized strictly diagonally dominant (GSDD) if there exist a vector $s = (s_i) > 0$ such that

$$|a_{ii}| s_i > \sum_{j \neq i}^{n} |a_{ij}| s_j.$$

This also means that there exists a diagonal matrix S such that $S^{-1}AS$ is strictly diagonally dominant; this is also true for AS.

In particular, strictly diagonally dominant and irreducibly diagonally dominant (cf. [33]) M-matrices and H-matrices are GSDD.

Now, to obtain bounds, simply consider the Jacobi iteration. Let $A = D + L + L^T$, D being the diagonal part and L the strictly lower triangular part of A,

$$x^k = -D^{-1}(L + L^T)x^{k-1} + D^{-1}e_j.$$

Let $J(A) = -D^{-1}(L + L^T)$ be the iteration matrix. Then, we have

$$e^k = J(A)^k e^0.$$

Using these definitions, the following result is obtained.

PROPOSITION 4.9. *If A is* GSDD, *we have*

$$s_i^{-1}|e_i^k| \le \|J(S^{-1}AS)^k\|_\infty \max_l [s_l^{-1}(A^{-1})_{lj}] \quad \forall j.$$

We also have the following result.

THEOREM 4.10. *If A is* GSDD,

$$(A^{-1})_{ij} \le s_i \|J(S^{-1}AS)^k\|_\infty \max_l [s_l^{-1}(A^{-1})_{lj}] \quad \forall i \notin S_j^k.$$

What follows is the problem of estimating $\|J(S^{-1}AS)^k\|_\infty$, which is strictly less than 1 as the Jacobi iteration is convergent for a GSDD matrix. Let $B = S^{-1}AS$ and ε, a given (small) positive real number; then we have the following proposition.

PROPOSITION 4.11. *For any matrix B and any $\varepsilon > 0$,*

$$\|B^k\|_\infty^{1/k} \le \rho(B) + \varepsilon,$$

where $\rho(B)$ is the spectral radius of B.

Proof. Let $B(\varepsilon) = \frac{1}{\rho(B)+\varepsilon} B$; then

$$\rho(B(\varepsilon)) < 1,$$

and $\lim_{k\to\infty} \|B^k(\varepsilon)\|_\infty = 0$. So, for k large enough, $\|B^k(\varepsilon)\|_\infty < 1$. As

$$\|B^k(\varepsilon)\|_\infty = \frac{\|B^k\|_\infty}{(\rho(B)+\varepsilon)^k},$$

the result is proved. $\square$

Now note that $\rho(J(S^{-1}AS)) = \rho(J(A))$. Therefore, the following theorem holds.

THEOREM 4.12. *If A is* GSDD, *we have*

$$(A^{-1})_{ij} \le s_i [\rho(J(A)) + \varepsilon]^k \max_l [s_l^{-1}(A^{-1})_{lj}] \quad \forall i \notin S_j^k.$$

The matrix S or the vector s can be chosen in many different ways. In fact, if A is an M-matrix and $y > 0$ is any given positive vector, we can get s by solving

$$As = y.$$

It is clear that by choosing y appropriately s can be made, for instance, close to a vector e made of 1s: there exists $\epsilon > 0$, such that $y = Ae + \epsilon > 0$; then $s = e + \epsilon A^{-1} e = e + \epsilon_A$.

THEOREM 4.13. *If A is a diagonally dominant M-matrix, there exists ε such that $\rho(J(A)) + \varepsilon < 1$ and*

$$\frac{(A^{-1})_{ij}}{(A^{-1})_{jj}} \le \frac{s_i}{1+\epsilon_A} [\rho(J(A)) + \varepsilon]^k \quad \forall i \notin S_j^k.$$

Proof. Use the previous result and Theorem 4.12. $\square$

Remark. If A is strictly diagonally dominant, the factor $(s_i/1 + \epsilon_A)$ can be replaced by 1.

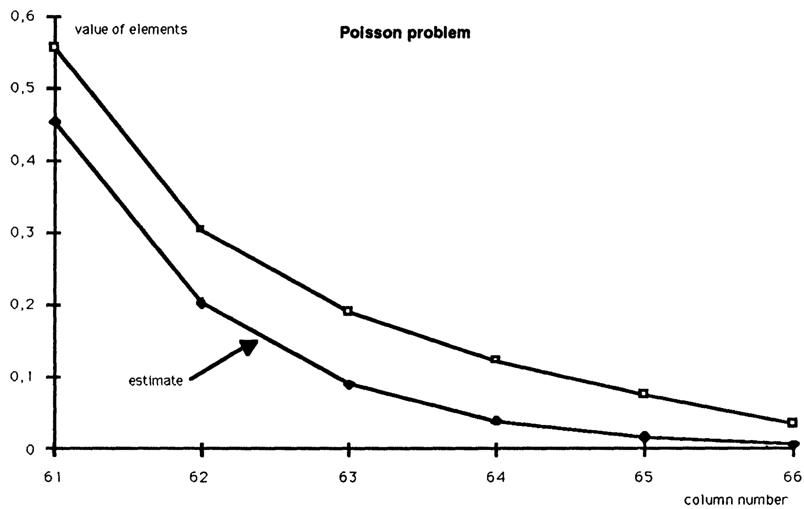

FIG. 6. *Comparison of the actual decay and the approximation.*

If we specialize to the Poisson problem, $\rho(J(A)) = \cos(\pi h) \simeq 1-(\pi^2h^2/2)$. Hence we get a better estimate than the one given in Theorem 4.5.

4.3. Approximation of the decay for general problems. For general block tridiagonal problems, the precise value of the condition number of the matrix is usually not known. The only information we have for some problems is that $\kappa = O(h^{-2})$. So, it is of interest to be able to compute a numerical approximation of the decay of the elements of the inverse. We can do this in the following way which mimics the INV preconditioner defined in [13]. Instead of computing the LU and UL decompositions as in §3.1, we are going to compute block incomplete factorizations.

Let $\mathrm{trid}(B)$ be a tridiagonal matrix with the nonzero elements that are the same as the corresponding ones in B. Then we define two incomplete block factorizations:

$$(\Delta + L)\ \Delta^{-1}\ (\Delta + L^T) \quad \text{and} \quad (\Sigma + L^T)\ \Sigma^{-1}\ (\Sigma + L),$$

where Δ and Σ are block diagonal matrices whose diagonal blocks are tridiagonal and denoted by Δ_i and Σ_i. They are given by the following formulas:

$$\begin{cases} \Delta_1 = D_1, \\ \Delta_i = D_i - A_i\ \mathrm{trid}[\Delta_{i-1}^{-1}]\ (A_i)^T, \end{cases} \qquad \begin{cases} \Sigma_n = D_n, \\ \Sigma_i = D_i - (A_{i+1})^T\ \mathrm{trid}[\Sigma_{i+1}^{-1}]\ A_{i+1}. \end{cases}$$

We then approximate the diagonal blocks of the inverse by the inverse of the tridiagonal matrix

$$D_j - A_j\ \mathrm{trid}[\Delta_{j-1}^{-1}]\ A_j^T - A_{j+1}^T\ \mathrm{trid}[\Sigma_{j+1}^{-1}]\ A_{j+1}.$$

When this tridiagonal matrix is computed and factored, we can obtain numerical information on the decay of the elements using the method of §2 and 3. This gives information on the decay along one direction of the two-dimensional mesh. To have information in the other direction (if the problem at hand is not isotropic), we can compute the other block elements with the formulas developed in §3.

Let us now compare the bounds obtained in this way with the actual decay of the elements on the Poisson problem. Figure 6 shows the actual decay of the elements in row 61 for a matrix on an 11×11 mesh and the decay of the estimate obtained in the previous way. It is seen that although the values are not very good, the behaviour of the curve is quite the same. However, this is a very simple example and this conclusion has to be checked on more general ones.

5. Conclusions. In this paper, we have exhibited useful relationships between the elements of inverses of tridiagonal and block tridiagonal matrices and elements of the Cholesky decompositions of these matrices. In particular, we got very simple expressions for the elements of the inverse of a block tridiagonal matrix. This allows us to develop stable algorithms for computing elements of the inverse when the matrix has more properties, like being diagonally dominant. The characterization of the inverse allows us also to obtain bounds for the decay of the elements of the inverse.

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