Review of preconditioners for Krylov methods

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KRYLOV METHODS

Aleksei N. Krylov, 1863–1945

Goal: approximately solve $Ax = b$, A large and sparse

x^0 given, initial residual $r^0 = b - Ax^0$

Krylov space $\mathcal{K}_k$ of order k based on A and r^0:

$$span\{r^0, Ar^0, \ldots, A^{k-1}r^0\}$$

Iterates:

$$x^k \in x^0 + \mathcal{K}_k$$

If V_k is a matrix whose columns are basis vectors $v^j, j = 1, \ldots, k$ of the Krylov space $\mathcal{K}_k$,

$$x^k = x^0 + V_k z^k, \quad r^k = r^0 - AV_k z^k$$

Two basic kinds of Krylov methods:

1) Orthogonal residual (OR) methods for which

$$(r^k)^T V_k = 0$$

-Conjugate Gradient (CG) for SPD matrices and FOM for general matrices

2) Minimum residual (MR) methods minimizes the l_2 norm of the residual $r^k = b - Ax^k$

$$(r^k)^T A V_k = 0$$

or solving a least squares problem

-MINRES for symmetric indefinite matrices and GMRES for general matrices

OR methods may break down for non SPD matrices

For stability reasons one uses an orthogonal basis of the Krylov space: **Arnoldi** process (1951) or **Lanczos** (1950) for symmetric matrices

Basis vectors are computed recursively by **Gram–Schmidt**

$$v^1 = \frac{r^0}{\|r^0\|}$$

The algorithm for computing column $j+1$ of V_k is

$$h_{i,j} = (Av^j, v^i), \quad i = 1, \ldots, j$$

$$\bar{v}^j = Av^j - \sum_{i=1}^{j} h_{i,j} v^i$$

$$h_{j+1,j} = \|\bar{v}^j\|$$

$$v^{j+1} = \frac{\bar{v}^j}{h_{j+1,j}}$$

Generally, one prefers using modified Gram–Schmidt (MGS)

$$w^j = Av^j,$$

and for $i = 1, \ldots, j$

$$h_{i,j} = (w^j, v^i), \quad w^j = w^j - h_{i,j} v^i$$

Both algorithms are the same in exact arithmetic, MGS is more stable but less parallel

$$AV_k = V_k H_k + h_{k+1,k} v^{k+1} (e^k)^T$$

H_k is an upper Hessenberg matrix with elements $h_{i,j}$

This can also be written as

$$AV_k = V_{k+1}\tilde{H}_k$$

with

$$\tilde{H}_k = \begin{pmatrix} H_k \\ h_{k+1,k}(e^k)^T \end{pmatrix}$$

Matrix $\tilde{H}_k$ is $(k+1) \times k$

We also have

$$V_k^T A V_k = H_k$$

H_k is the projection of A

GMRES (Generalized Minimum RESidual, Saad and Schulz 1986) minimizes the l_2 norm of the residual vector $b - Ax^k$, that is

$$\| \, \|r^0\| e^1 - \tilde{H}_k z^k \|$$

This is solved by incrementally computing a QR decomposition of $\tilde{H}_k$ using Givens rotations

In FOM we have to solve linear systems

$$H_k z^k = \|r^0\| e^1$$

Drawback: the storage grows with the iteration number since we have to store the basis vectors v^j

Solution: restart every m iterations $\rightarrow$ GMRES(m)

For GMRES if the symmetric part A_S of A is positive definite

$$\|r^k\| \leq \left(1 - \frac{\lambda_{min}(A_S)}{\sigma_{max}(A)}\right)^k \|r^0\|$$

where $\sigma_{max}(A) = \lambda_{max}(A^T A)$ is the largest singular value of A

For A SPD and CG

$$\|x - x^k\|_A^2 \leq 4 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^{2k} \|x - x^0\|_A^2$$

where $\kappa = \lambda_{max}/\lambda_{min}$ is the condition number of A

PRECONDITIONERS

Instead of

$$Ax = b$$

solve

$$M^{-1}Ax = M^{-1}b$$

or

$$M^{-\frac{1}{2}}AM^{-\frac{1}{2}}y = M^{-\frac{1}{2}}b$$

in the symmetric case

" M^{-1}" A is supposed to have better properties than A for convergence of the Krylov methods

Suppose A SPD large and sparse. Requirements on M:

- ◦ M SPD
- ◦ M sparse
- ◦ M easy and cheap to compute
- ◦ $Mz = r$ easy to solve
- ◦ "good" eigenvalue distribution for $M^{-1}A$
- • Constructing good preconditioners is more art than science
- ◦ Computing M must be parallel
- ◦ Solving $Mz = r$ must be parallel

Simplest idea: (**Jacobi**)

$$M = D = diag(A)$$

SSOR (Axelsson)

$$A = D + L + L^T$$

$$M = \frac{1}{\omega(2-\omega)}(D + \omega L)D^{-1}(D + \omega L^T)$$

For the Poisson equation, h=mesh size

$$\kappa(A) = O(\frac{1}{h^2})$$

$$\exists\ \omega_{opt}\ \text{tq}\ \kappa(M^{-1}A) = O(\frac{1}{h})$$

Pb: how to choose ω? In practice $\omega = 1$

Incomplete Cholesky decomposition

André–Louis Cholesky, 1875–1918

(. . . , Meijerink & Van der Vorst (1977),. . .)

The (complete) Cholesky outer product factorization

$$A = \bar{L}\bar{\Sigma}\bar{L}^T$$

$\bar{L}$ lower triang., $\bar{\Sigma}$ diagonal

The first step is

$$\bar{L}_1 = \begin{pmatrix} 1 & 0 \\ \bar{l}_1 & I \end{pmatrix}, \quad \bar{\Sigma}_1 = \begin{pmatrix} a_{1,1} & 0 \\ 0 & \bar{A}_2 \end{pmatrix}$$

and

$$A = \begin{pmatrix} a_{1,1} & \bar{a}_1^T \\ \bar{a}_1 & \bar{B}_1 \end{pmatrix} = \bar{L}_1\bar{\Sigma}_1 L_1^T$$

$$\bar{l}_1 = \frac{\bar{a}_1}{a_{1,1}}$$

$$\bar{A}_2 = \bar{B}_1 - \frac{1}{a_{1,1}} \bar{a}_1 \bar{a}_1^T$$

$$\bar{A}_2 = \begin{pmatrix} \bar{a}_{2,2}^{(2)} & \bar{a}_2^T \\ \bar{a}_2 & \bar{B}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{l}_2 & I \end{pmatrix} \begin{pmatrix} \bar{a}_{2,2}^{(2)} & 0 \\ 0 & \bar{A}_3 \end{pmatrix} \begin{pmatrix} 1 & \bar{l}_2^T \\ 0 & I \end{pmatrix}$$

$$\bar{L}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \bar{l}_2 & I \end{pmatrix}$$

$$\bar{\Sigma}_1 = \begin{pmatrix} a_{1,1} & 0 \\ 0 & \bar{A}_2 \end{pmatrix} = \bar{L}_2 \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & \bar{a}_{2,2}^{(2)} & 0 \\ 0 & 0 & \bar{A}_3 \end{pmatrix} \bar{L}_2^T = \bar{L}_2 \bar{\Sigma}_2 \bar{L}_2^T$$

After two steps, we have $A = \bar{L}_1 \bar{L}_2 \bar{\Sigma}_2 \bar{L}_2^T \bar{L}_1^T$

$$\bar{L}_1 \bar{L}_2 = \begin{pmatrix} 1 & 0 \\ \bar{l}_1 & \begin{pmatrix} 1 & 0 \\ \bar{l}_2 & I \end{pmatrix} \end{pmatrix}$$

Finally, if all the pivots are non–zero

$$A = \bar{L}_1 \cdots \bar{L}_{n-1} \bar{\Sigma} \bar{L}_{n-1}^T \cdots \bar{L}_1^T$$

This works if A is SPD

Incomplete Cholesky

Let $G = \{(i,j),\ i > j\}$ be a given set of indices

$$A = L\Sigma L^T - R$$

L being lower triangular with $l_{i,i} = 1$ and Σ diagonal

Construct L s.t. that $l_{i,j} = 0$ if $(i,j) \notin G$

$$A = A_1 = \begin{pmatrix} a_{1,1} & a_1^T \\ a_1 & B_1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & b_1^T \\ b_1 & B_1 \end{pmatrix} - \begin{pmatrix} 0 & r_1^T \\ r_1 & 0 \end{pmatrix} = M_1 - R_1$$

$$a_1 = b_1 - r_1$$

$$(b_1)_i = 0, \text{ if } (i,1) \notin G \Rightarrow (r_1)_i = -(a_1)_i$$

$$(b_1)_i = (a_1)_i, \text{ if } (i,1) \in G \Rightarrow (r_1)_i = 0$$

Then,

$$M_1 = \begin{pmatrix} 1 & 0 \\ l_1 & I \end{pmatrix} \begin{pmatrix} a_{1,1} & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} 1 & l_1^T \\ 0 & I \end{pmatrix} = L_1 \Sigma_1 L_1^T$$

$$l_1 = \frac{b_1}{a_{1,1}}$$

$$A_2 = B_2 - \frac{1}{a_{1,1}} b_1 b_1^T$$

We use the same process on A_2

$$A_2 = \begin{pmatrix} a_{2,2}^{(2)} & a_2^T \\ a_1 2 & B_2 \end{pmatrix} = \begin{pmatrix} a_{2,2}^{(2)} & b_2^T \\ b_2 & B_2 \end{pmatrix} - \begin{pmatrix} 0 & r_2^T \\ r_2 & 0 \end{pmatrix} = M_2 - R_2$$

where b_2 is obtained from a_2 by zeroing the elements $(i, 2)$ whose indices do not belong to G

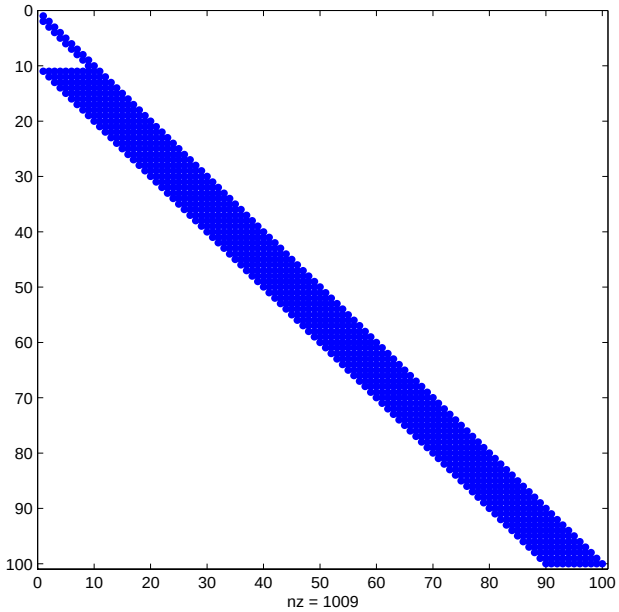
$$L_2 = \begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ l_2 & I \end{pmatrix} \end{pmatrix}$$

$$A = L_1 L_2 \Sigma_2 L_2^T L_1^T - L_1 \begin{pmatrix} 0 & 0 \\ 0 & R_2 \end{pmatrix} L_1^T - R_1$$

$$L_1 L_2 = \begin{pmatrix} 1 & 0 \\ l_1 & \begin{pmatrix} 1 & 0 \\ l_2 & I \end{pmatrix} \end{pmatrix} \quad \text{and } L_1 \begin{pmatrix} 0 & 0 \\ 0 & R_2 \end{pmatrix} L_1^T = \begin{pmatrix} 0 & 0 \\ 0 & R_2 \end{pmatrix}$$

This algorithm has constructed a complete decomposition of $M = A + R$

For Poisson–like equations



In the Cholesky decomposition of A, $A = \tilde{L}\tilde{L}^T$, we get **fill–in**

Incomplete Cholesky (IC) is easy for 5 diagonal matrices

$$M = L \; D^{-1} \; L^T$$

If

$$A = (c_i, b_i, a_i, b_{i+1}, c_{i+m})$$

$$L = (\bar{c}_i, \bar{b}_i, d_i, 0, 0), \;\; D = (0, 0, d_i, 0, 0)$$

Then,

$$\bar{c}_i = c_i$$

$$\bar{b}_i = b_i$$

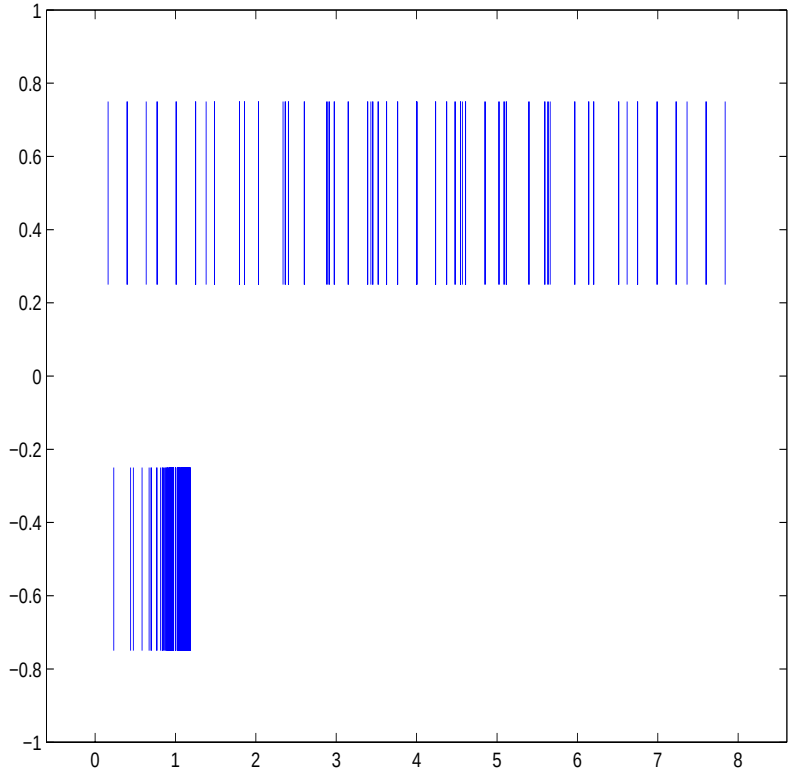
$$d_i = a_i - \frac{b_i^2}{d_{i-1}} - \frac{c_i^2}{d_{i-m}}$$

This is **IC(1,1)**

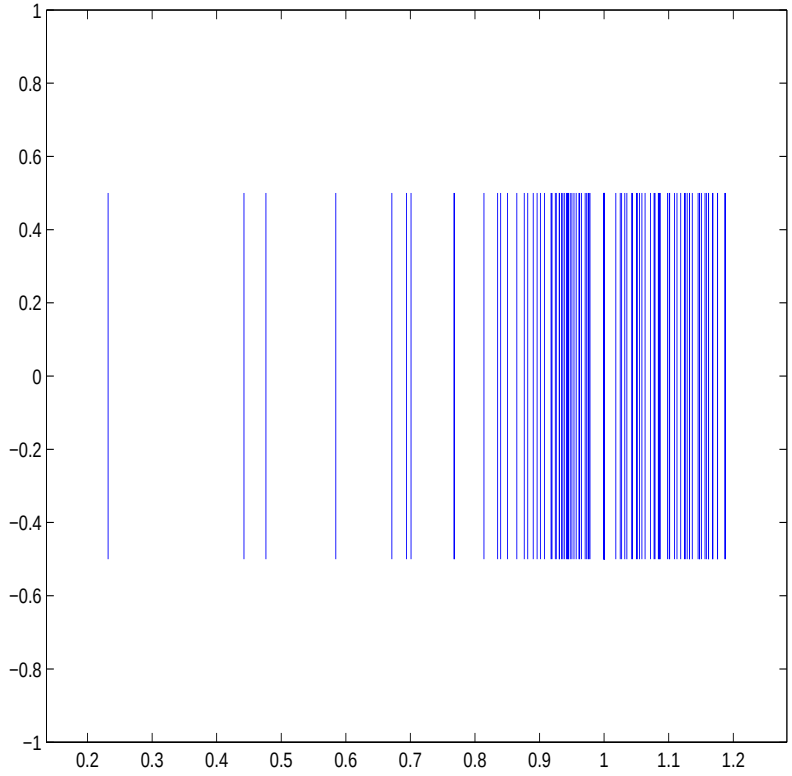
For the Poisson equation we still have

$$\kappa(M^{-1}A) = O(\frac{1}{h^2})$$

but a "good" distribution of eigenvalues



Poisson, $m \times m$ mesh, $m = 10$, A and $IC(1,1)$ on A



Poisson, $m \times m$ mesh, $m = 10$, $IC(1,1)$

We can use other orderings of the unknowns

$$A_P = PAP^T$$

Orderings can have a very large impact on convergence

- Lichnewsky 1984 (nested dissection)
- Simon 1985
- Duff–Meurant 1989 (numerical experiments)

Theoretical explanation:

- V. Eijkhout 1990
- S. Doi 1990

$$M = LDL^T = A + R$$

Ordering examples

- ROW (row)
- CM (Cuthill–Mc Kee)
- MIND (Minimum degree)
- RB (Red–Black)
- ND (Nested dissection)
- VDV2 (Van der Vorst)

Table 1: Poisson 30× 30 mesh

ordering	nit	nb of fills	nb R	$\|R\|_F^2$
ROW	23	24389	841	142.5
CM	23	16675	841	142.5
MIND	39	7971	1582	467.3
RB	38	12853	1681	525.5
ND	25	15228	1012	157.1
VDV2	20	17413	841	140.7

nb elements in L : 2639

Table 2: Anisotropic problem $a = 100, b = 1$, 30 $\times$ 30 mesh

ordering	nit	nb of fills	nb R	$\|R\|_F^2$
ROW	9	24389	841	$0.12\ 10^4$
CM	9	16675	841	$0.12\ 10^4$
MIND	48	7971	1582	$0.18\ 10^7$
RB	47	12853	1681	$0.21\ 10^7$
ND	26	15228	1012	$0.43\ 10^6$
VDV2	9	17413	841	$0.11\ 10^4$

Doi et Eijkhout's theory explain these results

Situation can different if we keep some fills

Modified preconditioners

(Dupont, Kendall & Rachford)

$$M = LD^{-1}L^T = A + R$$

Modify D s.t. $rowsum(R) = 0$ or ch^2

For a 5 diagonal matrix:

$$d_i = (1 + ch^2)a_i - \frac{b_{i-1}(b_{i-1} + c_{i-1})}{d_{i-1}} - \frac{c_{i-m}(c_{i-m} + b_{i-m})}{d_{i-m}}$$

For Poisson

$$\kappa(M^{-1}A) = O(\frac{1}{h})$$

instead of $O(1/h^2)$

Block preconditioners

(Concus, Golub & Meurant 1985, Axelsson)

$$A = \begin{pmatrix} D_1 & A_2^T & & & \\ A_2 & D_2 & A_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1} & D_{n-1} & A_n^T \\ & & & A_n & D_n \end{pmatrix}$$

D_i tridiagonal, A_i diagonal

$$L = \begin{pmatrix} 0 & & & \\ A_2 & 0 & & \\ & \ddots & \ddots & \\ & & A_n & 0 \end{pmatrix}$$

Block Cholesky is

$$A = (\Sigma + L)\ \Sigma^{-1}\ (\Sigma + L^T)$$

$$\Sigma = \begin{pmatrix} \Sigma_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Sigma_n \end{pmatrix}$$

$$\begin{cases} \Sigma_1 = D_1 \\ \Sigma_i = D_i - A_i\ \Sigma_{i-1}^{-1}\ {A_i}^T \end{cases}$$

INV (Block incomplete Cholesky)

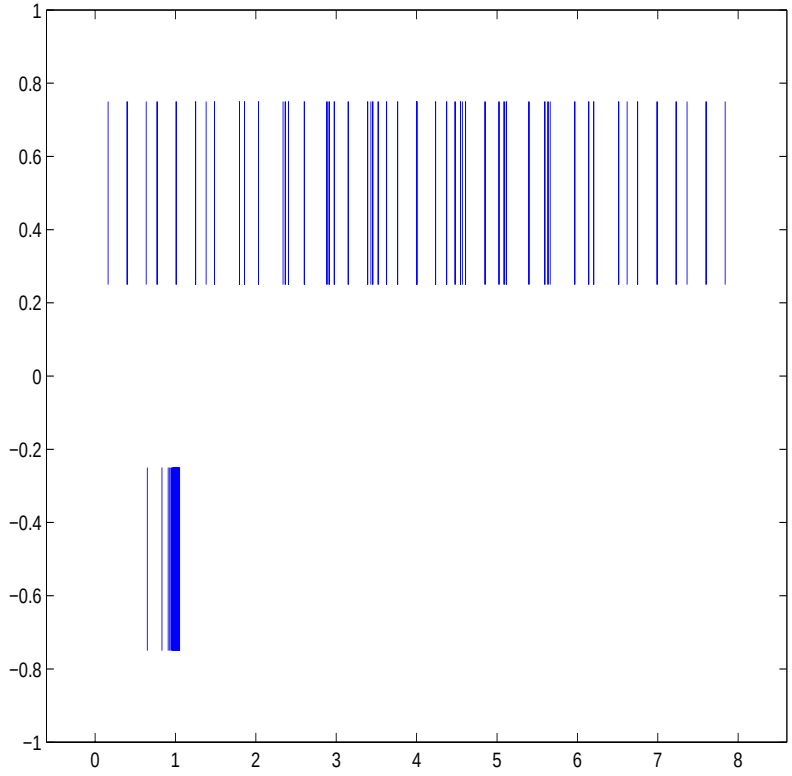
$$A = (\Delta + L)\ \Delta^{-1}\ (\Delta + L^T)$$

$$\Delta = \begin{pmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Delta_n \end{pmatrix}$$

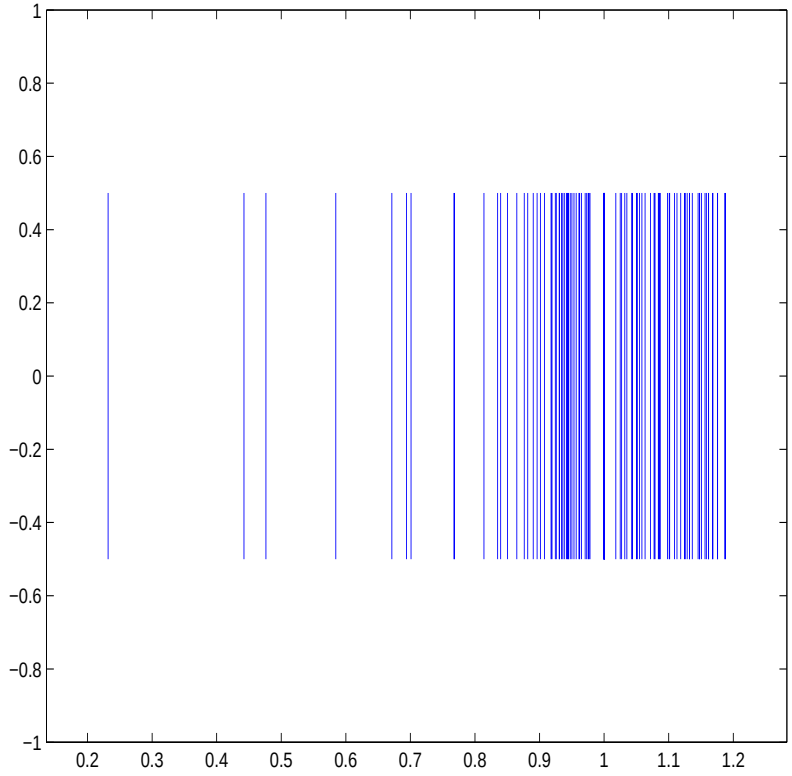
$$\begin{cases} \Delta_1 = D_1 \\ \Delta_i = D_i - A_i\ trid(\Delta_{i-1}^{-1})\ {A_i}^T \end{cases}$$

Δ_i tridiagonal

INV is feasible for H–matrices



Poisson, $m \times m$ mesh, $m = 10$, A and $INV(A)$



Poisson, $m \times m$ mesh, $m = 10$, $INV(A)$

Polynomial preconditioners

$$M^{-1} = P_k(A) = \sum_{i=0}^{k} \alpha_i A^i$$

P_k polynomial of degree k

Eigenvalues of $M^{-1}A$ are $P_k(\lambda_i)\lambda_i$

We would like $P_k(\lambda)\lambda$ to be close to 1 on $[\lambda_{min}, \lambda_{max}] \subset [a, b]$

Interest: the solve $Mz^k = r^k$ is parallel

MINMAX polynomial

(Johnson, Michelli & Paul, 1983)

$$q_{k+1}(\lambda) = p_k(\lambda)\lambda$$

$$\mathcal{Q}_k = \{ \text{ polynomials of deg } k, \text{ positive, value 0 at 0 } \}$$

Eigenvalues of A in $[a, b]$, minimize over $\mathcal{Q}_k$

$$cond(q) = \frac{sup_{\lambda \in [a,b]} q_{k+1}(\lambda)}{inf_{\lambda \in [a,b]} q_{k+1}(\lambda)}$$

Solution:

$$q_{k+1}(\lambda) = 1 - \frac{C_{k+1}(\mu(\lambda))}{C_{k+1}(\mu(0))}$$

with $\mu(\lambda) = \frac{2\lambda - b - a}{b - a}$ and C_k Chebychev polynomial

Least squares polynomial

(Saad, 1985)

Look for s, polynomial which minimizes

$$\int_a^b (1 - \lambda s(\lambda))^2 w(\lambda)\ d\lambda$$

w is a chosen weight

Usual choice:

$$w(\lambda) = (b - \lambda)^\alpha (\lambda - a)^\beta$$

$\alpha \geq \beta \geq -1/2$: Jacobi polynomial

In practice $\alpha = \beta = -\frac{1}{2}$ (Chebychev) or $\alpha = \beta = 0$ (Legendre)

Drawbacks:

◦ we need a and b

◦ when the degree is large ($\geq$ 10 ou 20) applying the polynomial can be difficult because of rounding errors – instability of Horner's scheme

• Use recurrences (orthogonal polynomials)

Example for Minmax,

$$p_k(\lambda) = \frac{4}{a-b}\frac{c_k}{c_{k+1}} + 2\mu(\lambda)\frac{c_k}{c_{k+1}}p_{k-1}(\lambda) - \frac{c_{k-1}k}{c_{k+1}}p_{k-2}(\lambda)$$

$$p_0(\lambda) = \frac{2}{a+b}, \quad p_1(\lambda) = \frac{8(a+b-\lambda)}{a^2+b^2+6ab}$$

$$c_k = C_k(\mu(0))$$

Problem: more costly

Approximate inverses

- Huckle et Grote (1994)
- Gould et Scott (1995)
- Chow et Saad (1994–1995)
- Benzi (1995–1996)

We want $M^{-1}A$ "to look like" I

Define directly M^{-1}

We compute $C = M^{-1}$ to minimize

$$\|AC - I\| \text{ or } \|CA - I\|$$

Generally one takes the Frobenius norm:

$$\|AC - I\|_F^2 = \sum_{k=1}^{n} \|(AC - I)e_k\|^2,$$

e_k k–th column of I, we minimize the l_2 norms

$$\|Ac_k - e_k\|, \ k = 1, \ldots, n$$

n independent least squares problems (parallel)

Generally A^{-1} is dense, how to choose the sparsity structure of c_k?

Let $\hat{c}_k$ be the vector of the non zero elements of c_k

Let $\hat{A}_k$ be the matrix whose columns are those of A with indices $G_k = \{j | (c_k)_j \neq 0\}$ and whose rows i are such that $\exists\ a_{i,j} \neq 0, j \in G_k$

$$\min_{\hat{c}_k} \|\hat{A}_k \hat{c}_k - \hat{e}_k\|, \quad k = 1, \ldots, n$$

These small least squares problems are solved with QR

Structure of c_k:

algorithm of Huckle & Grote starting from the diagonal or A

This method is denoted as SPAI

Parallel implementation was considered by Deshpande, Grote, Messmer and Sawyer, 1996

Gould and Scott, 1995 improved the choice of the structure

Benzi, Meyer and Tuma approximate inverse

A SPD

If $Z = [z_1, z_2, \ldots, z_n]$ is a set of conjugate directions for A

$$Z^T A Z = D$$

D diagonal and $A^{-1} = Z D^{-1} Z^T$

The directions are computed by Gram–Schmidt applied to $v_1, v_2, \ldots, v_n$

If $V = [v_1, v_2, \ldots, v_n] = I$, Z is upper triangular

1) $z_i^{(0)} = e_i, \quad i = 1, \ldots, n$

2) for $i = 1, \ldots, n \quad d_j^{(i-1)} = (a_i, z_j^{(i-1)}), \quad j = i, \ldots, n$ where a_i is the ith column of A

if $j \neq n$, $z_j^{(i)} = z_j^{(i-1)} - \left(\frac{d_j^{(i-1)}}{d_i^{(i-1)}} \right) z_i^{(i-1)}, \quad j = i+1, \ldots, n$

3) $z_i = z_i^{(i-1)}, d_i = d_i^{(i-1)}, \quad i = 1, \ldots, n$

To preserve the sparsity structure, fills are thrown away based on position or value (or both)

This method is known as AINV

Benzi, Meyer and Tuma, 1996 show that this method is feasible for H–matrices

There exists a "robust" variant SAINV (Benzi, Cullum and Tuma)

This is generalized to non symmetric matrices by considering two matrices $Z = [z_1, \ldots, z_n]$ and $W = [w_1, \ldots, w_n]$ such that

$$W^T A Z = D$$

Comparison between IC and AINV (Benzi)

Table 3: Poisson, L-shaped region, mixed b.c.

IC			AINV		
fill	nb. iter	time	fill	nb. iter	time
675	87	0.33	743	76	0.32
897	53	0.18	780	74	0.32
912	51	0.18	1135	54	0.26
1204	38	0.14	1208	47	0.18
1439	32	0.14	1300	40	0.21
1565	24	0.10	3654	22	0.14

Table 4: Comparison between SPAI and AINV (Benzi)

Matrix	SPAI			AINV		
	Its	init	t its	Its	init	t its
3DCD	40	10.63	0.111	25	1.885	0.068
ALE3D	45	30.79	0.088	43	1.446	0.094
ORSREG1	40	3.309	0.033	33	0.550	0.031
SHERMAN1	62	0.878	0.029	43	0.201	0.021
PORES3	111	0.941	0.044	75	0.127	0.038
WATT2	377	2.590	0.384	111	0.505	0.116

MULTILEVEL PRECONDITIONERS

◦ Algebraic methods (grid $\equiv$ (sub) set of unknowns)

◦ Algebraic MultiGrid (AMG)–like (V–cycle): Starting from the zero vector:

0– on coarsest level, solve exactly by Gaussian elimination, otherwise

1– do ν iterations of smoothing

2– restrict the residual r to $r_c = Rr$

3– recursively solve $A_c e_c = r_c$, $A_c = RAP$, $R = P^T$

4– interpolate e_c to $e = Pe_c$

5– add the correction e to the current iterate

6– do ν iterations of smoothing

We have to define

$\circ$ the smoother

$\rightarrow$ $\circ$ how to construct the coarse grids (coarsening alg.)

$\rightarrow$ $\circ$ the interpolation

Smoothers

- Symmetric Gauss–Seidel (not naturally //)
- Incomplete Cholesky (not // either) LDL^T
  - IC(0)
  - IC with some fill–in on values
  - IC with some fill–in on levels of fill

$$LD^{-1}L^T(x^{k+1} - x^k) = b - Ax^k$$

- Approximate inverse AINV from M. Benzi and al.

The influence matrix

$$\mathcal{N} = \{1, \ldots, n\} = F \cup C$$

standard AMG choice for M–matrices

$$S_i = \{j | -a_{i,j} > \theta \max_{k \neq i}(-a_{i,k}), \quad \theta < 1\}$$

This gives S (by padding with zeros)

Generalization:

$$S_i^A = \{j \mid |a_{i,j}| > \tau \max_{k \neq i} |a_{i,k}|, \quad \tau < 1\}$$

The coarsening algorithm

Weights w_i = the number of points that depend on i (using S)

1- choose the first point i with maximal weight as a C point

2- assign the points that i influences (using S) as F points

3- increment by 1 the weights of the points influencing these new F points (to give more chances to be selected as C points)

4- decrease by 1 the weights of points that depends on i

repeat from step 1- until all points are labeled

The interpolation algorithm

$$\omega_{i,j} = -\frac{a_{i,j} + \sum_{k \in D_i^S} \frac{a_{i,k} a_{k,j}}{\sum_{m \epsilon C_i} a_{k,m}}}{a_{i,i} + \sum_{k \in D_i^W} a_{i,k}}$$

Note that the given F point needs to have at least one coarse point in its neighborhood in the graph of A

The coarse matrices

The interpolation algorithm defines P and $R = P^T$

$$A_C = RAP$$

How to parallelize the smoothers?

◦ partition the graph of A into subdomains and interfaces or with overlapping (ghost nodes)

◦ Symmetric Gauss–Seidel

parallelized by using Jacobi for the interface nodes (SGSJ)

◦ Incomplete Cholesky or AINV

parallelized by ignoring dependencies between subdomains (ICp, SAINVp)

We partition only the finest level graph

Parallel coarsening

- LLNL algorithms (Cleary, Falgout, Henson and Jones)

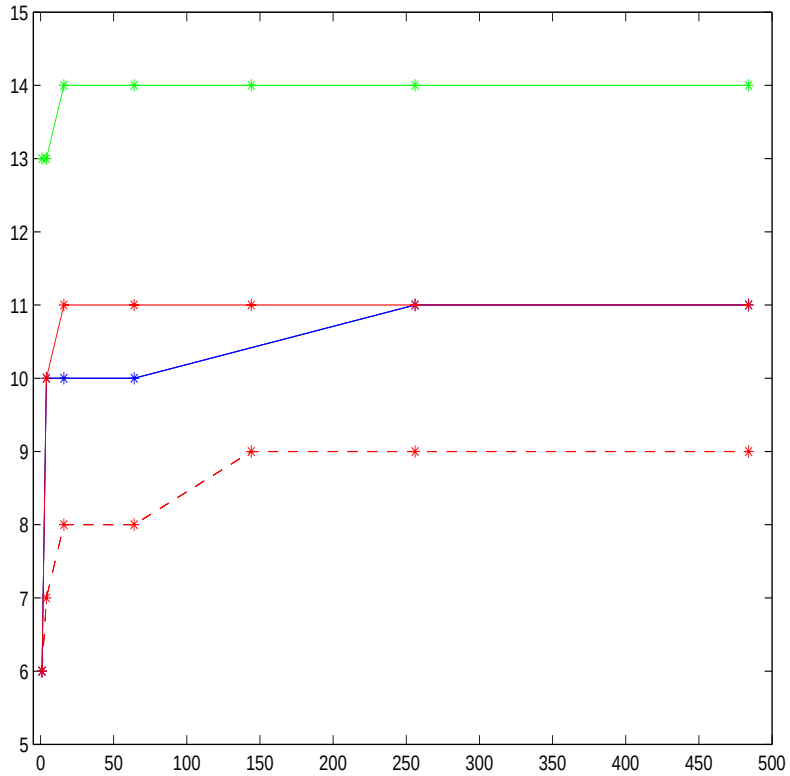
The CEA teraflops parallel computer

- HP COMPAQ SMP
- 640 nodes, 2560 processors Alpha EV68 1Ghz
- Memory 4GB per node
- Quadrics interconnection network
- in production since end of 2002

The CEA Teraflops machine

First experiment

- Poisson $m_p = 250 \rightarrow 62500$ unknowns per processor
- Number of proc: $p = 1, 4, 16, 64, 144, 256, 484$
- Largest problem is $\simeq 30\ 10^6$ unknowns
- May be a little too large for a 2D problem!



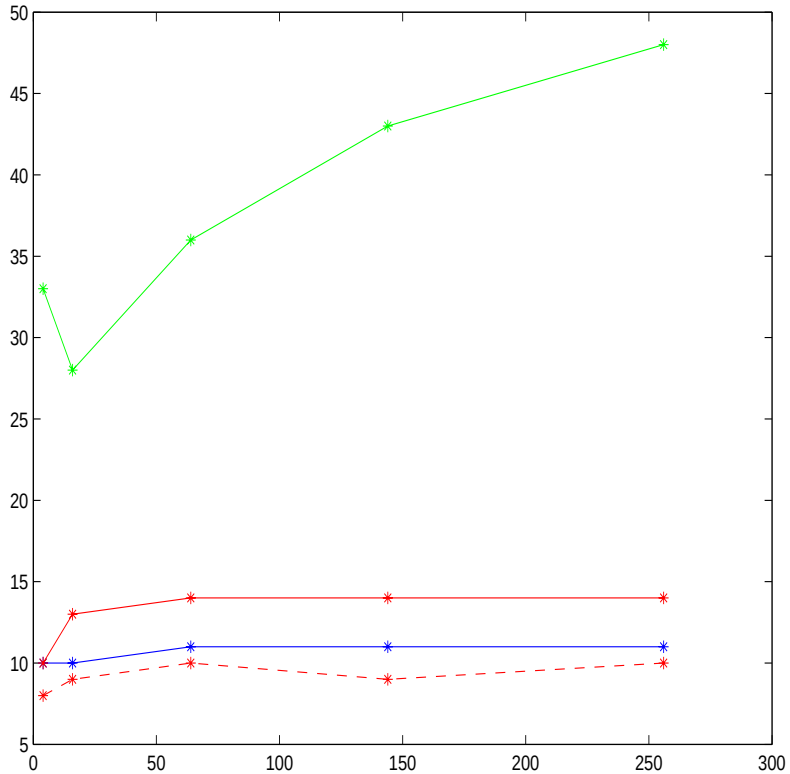
Nb of iterations for the Poisson equation as a function of p,coarsening: LLNL, blue: SGSJ, red dashed: IC, red: ICp, green: SAINVp

Second experiment

- Discontinuous problem

$m_p = 250 \rightarrow 62500$ unknowns per processor

- Number of proc: $p = 4, 16, 64, 144, 256, 484$
- Largest problem is $\simeq 30\ 10^6$ unknowns



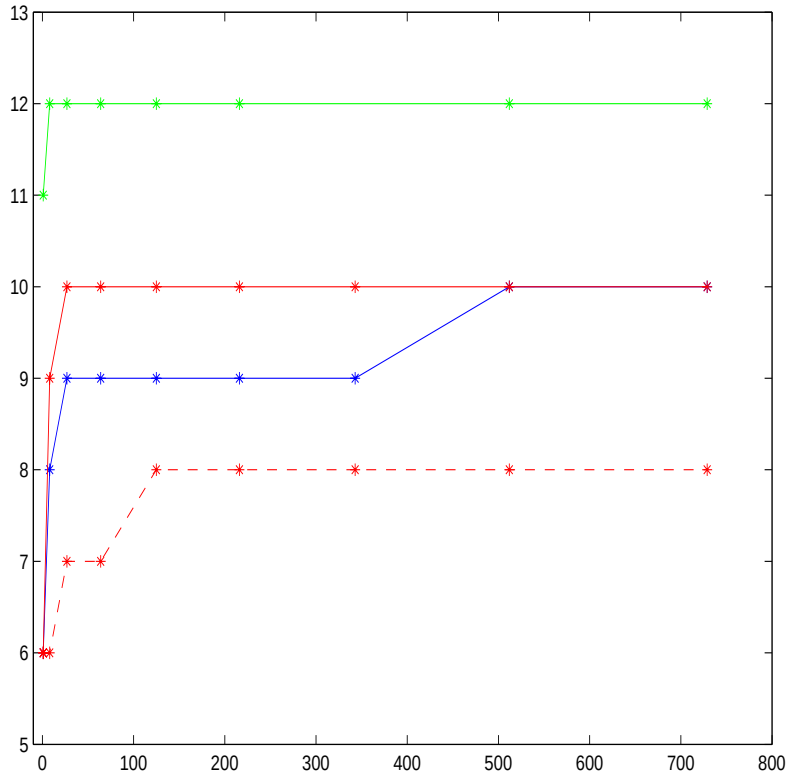
Nb of iterations for the discontinuous problem as a function of p,coarsening: LLNL, blue: SGSJ, red dashed: IC, red: ICp, green: SAINVp

Last experiment

- 3D Poisson in $[0,1]^3$

$m_p = 30 \rightarrow m_p^3 = 27000$ unknowns per processor

- Number of proc: $p = 1, 8, 27, 64, 125, 216, 343, 512, 729$
- Largest problem is $\simeq 20\ 10^6$ unknowns



Nb of iterations for the 3D Poisson equation as a function of p,coarsening: LLNL, blue: SGSJ, red dashed: IC, red: ICp, green: SAINVp

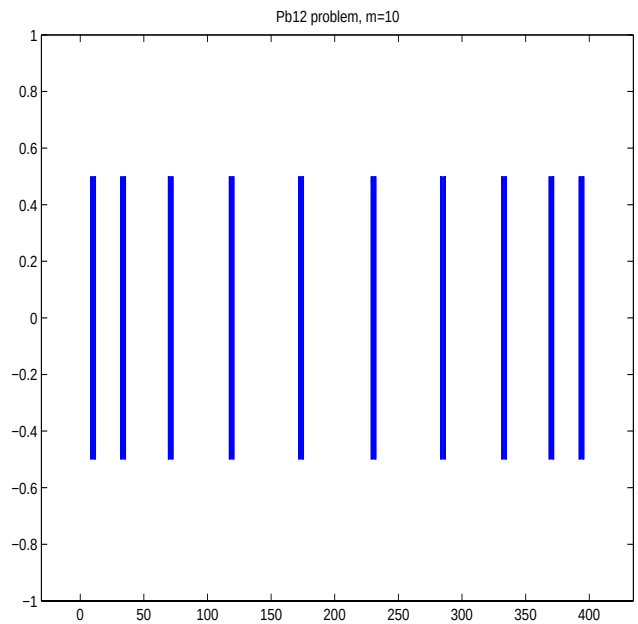
Numerical Experiments

SPD matrices with CG

◦ Anisotropic problem (Pb12)

x–diffusion coefficient$= 1$, y–diffusion coefficient$= 100$

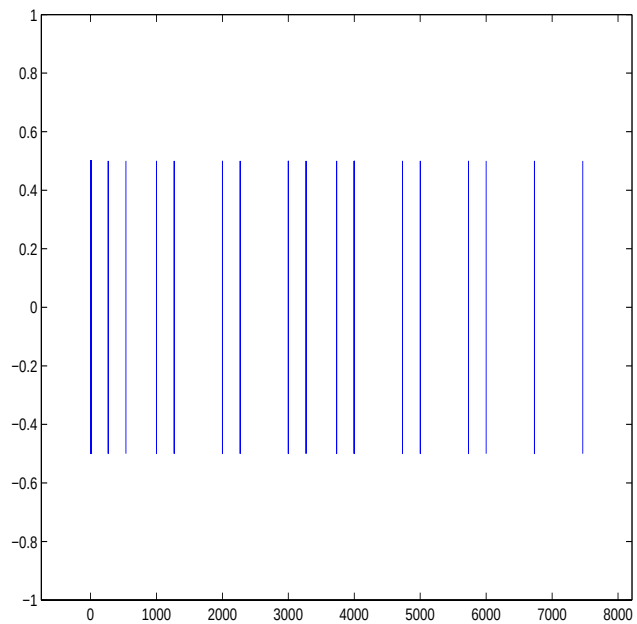
for $m = 50$, $\lambda_{min} = 0.3831$, $\lambda_{max} = 4.04\ 10^2$, $\kappa = 1.05\ 10^3$



◦ A discontinuous problem (Pb14)

x and y –diffusion coefficients are 1 except in the square $[1/4, 3/4]^2$ where the value is 1000

for $m = 50$, $\lambda_{min} = 8.82\ 10^{-3}$, $\lambda_{max} = 7.97\ 10^3$, $\kappa = 9.03\ 10^5$



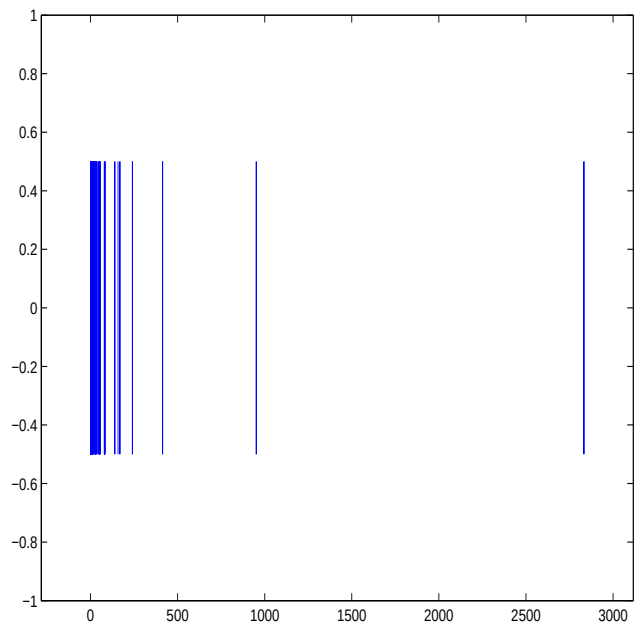
○ The sine–sine problem (Pb26)

x and y–coefficients are equal. The diffusion coefficient is

$$\frac{1}{(2+p\sin\frac{x}{\eta})(2+p\sin\frac{y}{\eta})}$$

where $p = 1.99$ and $\eta = 0.01$

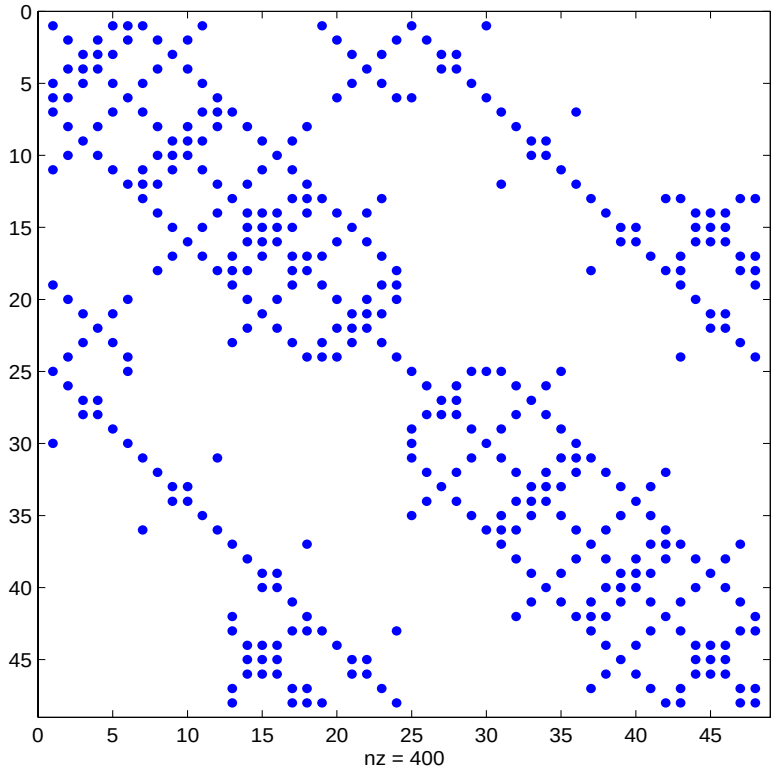
for $m = 50$, $\lambda_{min} = 1.81\ 10^{-2}$, $\lambda_{max} = 8.86\ 10^{3}$, $\kappa = 4.89\ 10^{5}$



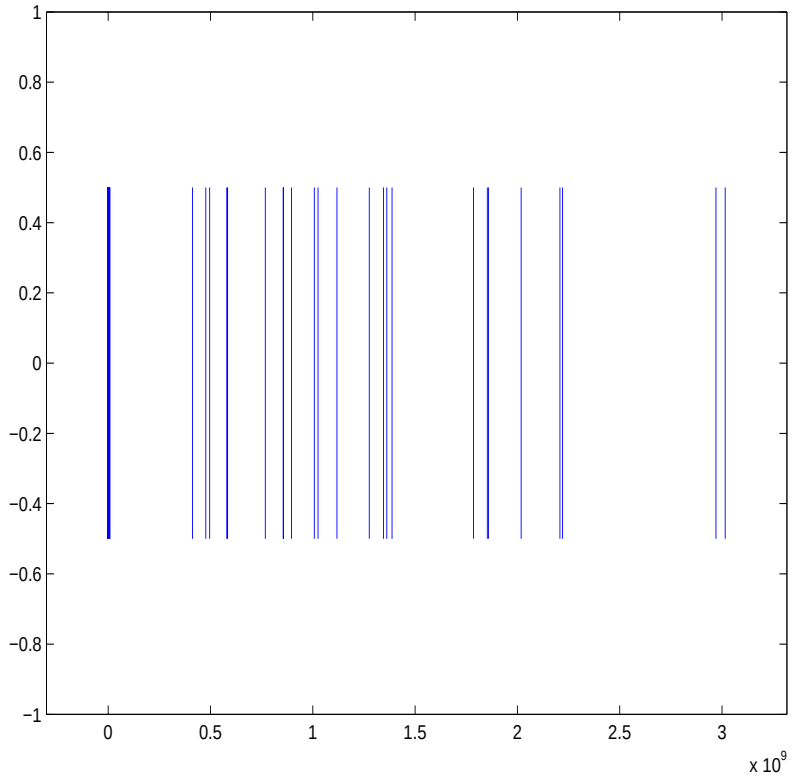
Matrices from the Matrix Market

http://math.nist.gov

- Bcsstk01 $n = 48$

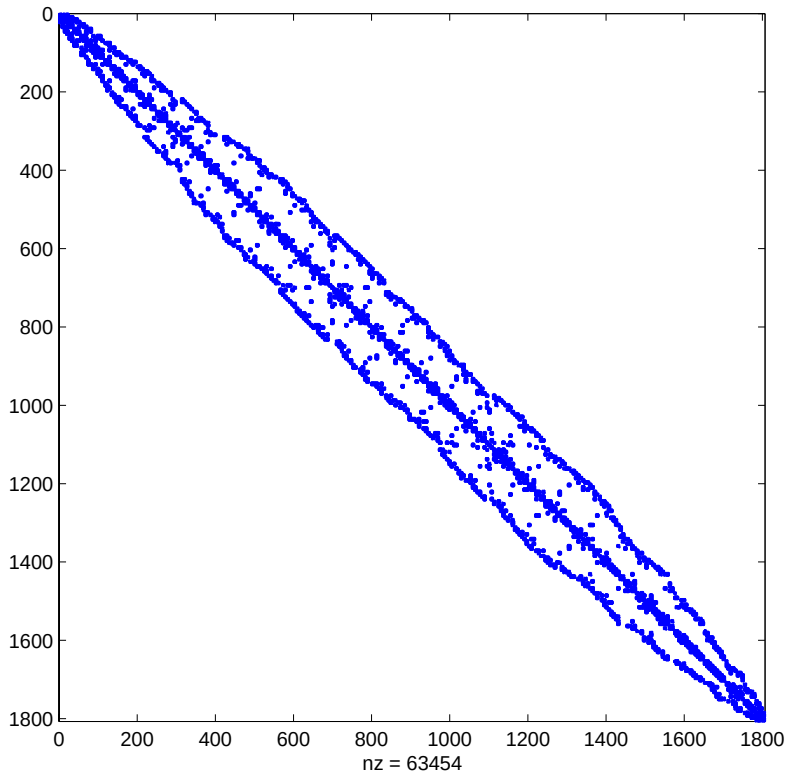


$$\lambda_{min} = 3.42\ 10^3, \quad \lambda_{max} = 3.02\ 10^9,\ \kappa(A) = 8.82\ 10^5$$

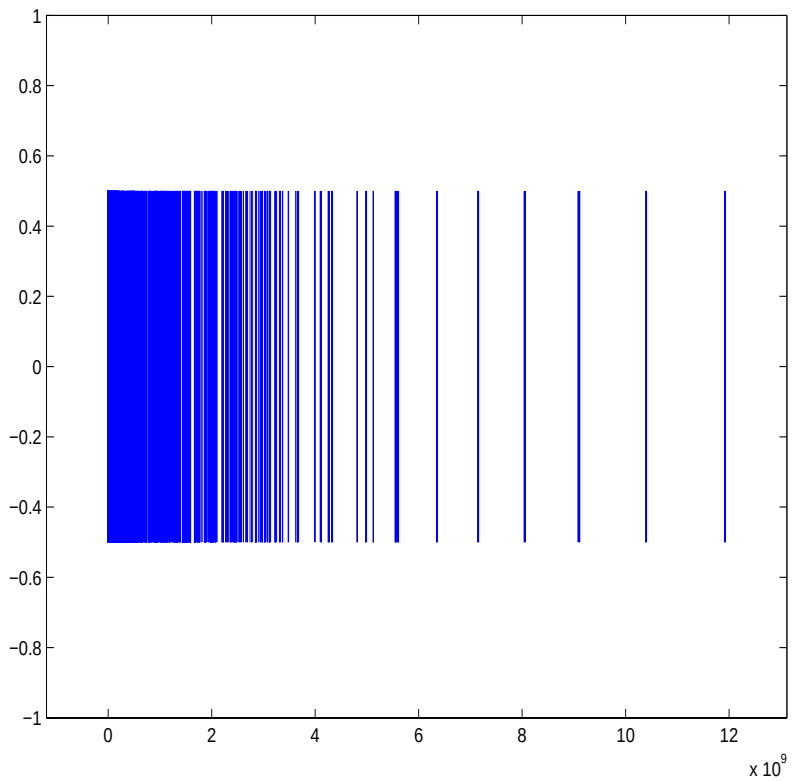


Eigenvalues of Bcsstk01

- Bcsstk14 $n = 1806$

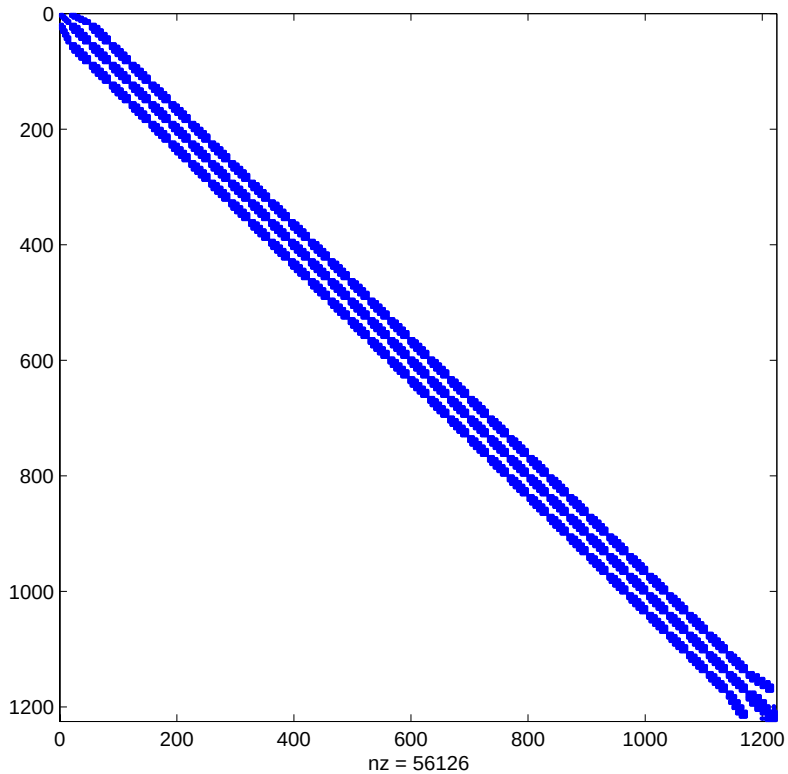


$$\lambda_{min} = 1, \quad \lambda_{max} = 1.19\ 10^{10},\ \kappa = 1.19\ 10^{10}$$

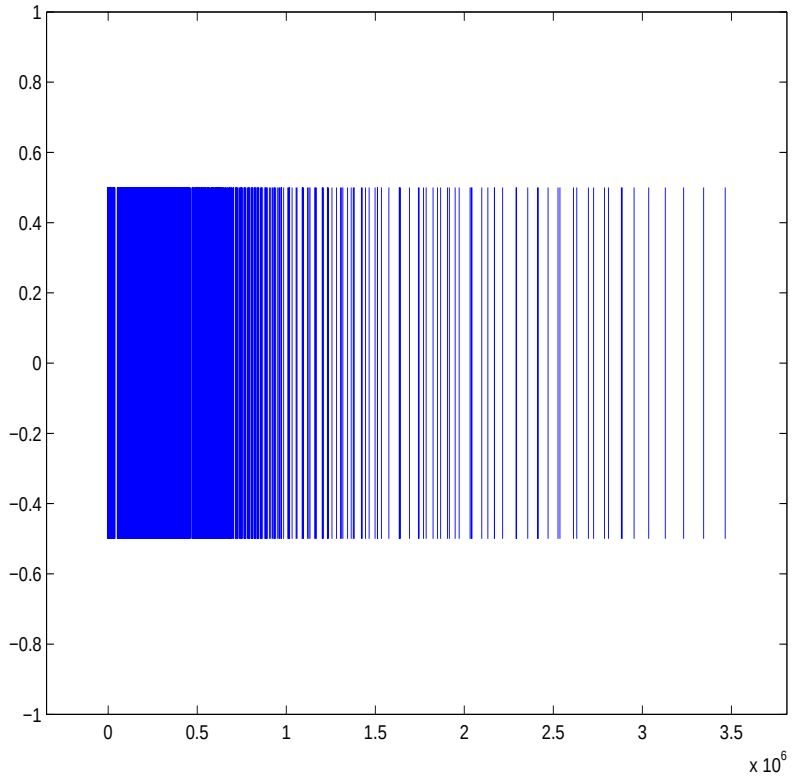


Eigenvalues of Bcsstk14

- Bcsstk27 $n = 1224$

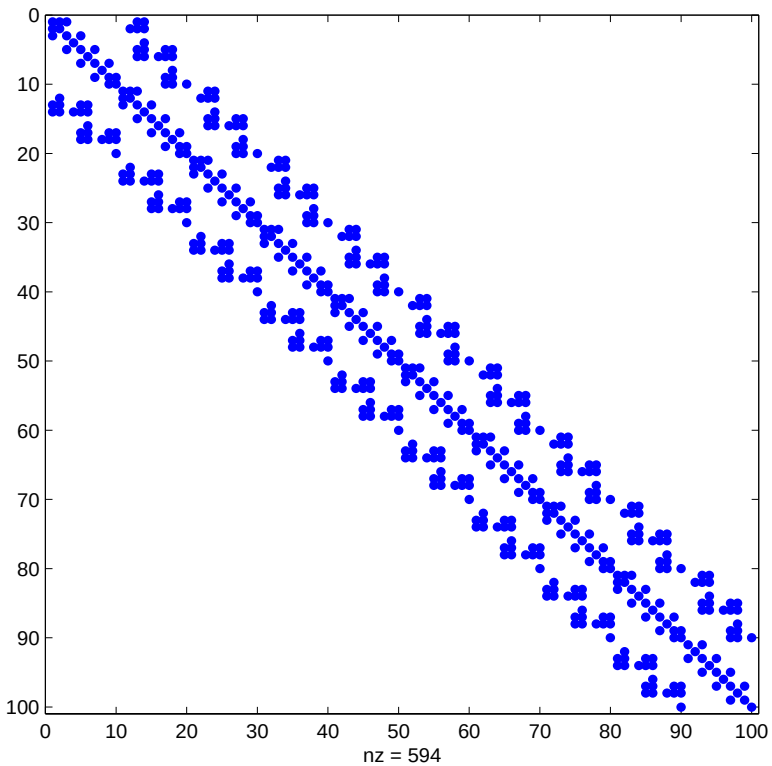


$$\lambda_{min} = 1.44\ 10^2, \quad \lambda_{max} = 3.46\ 10^6,\ \kappa(A) = 2.41\ 10^4$$

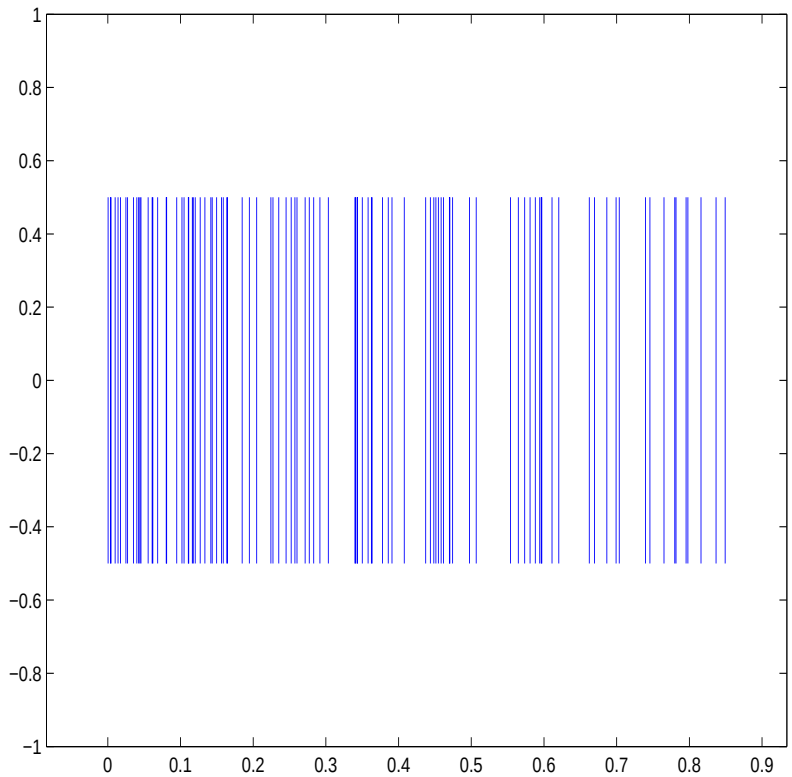


Eigenvalues of Bcsstk27

- Nos4 $n = 100$

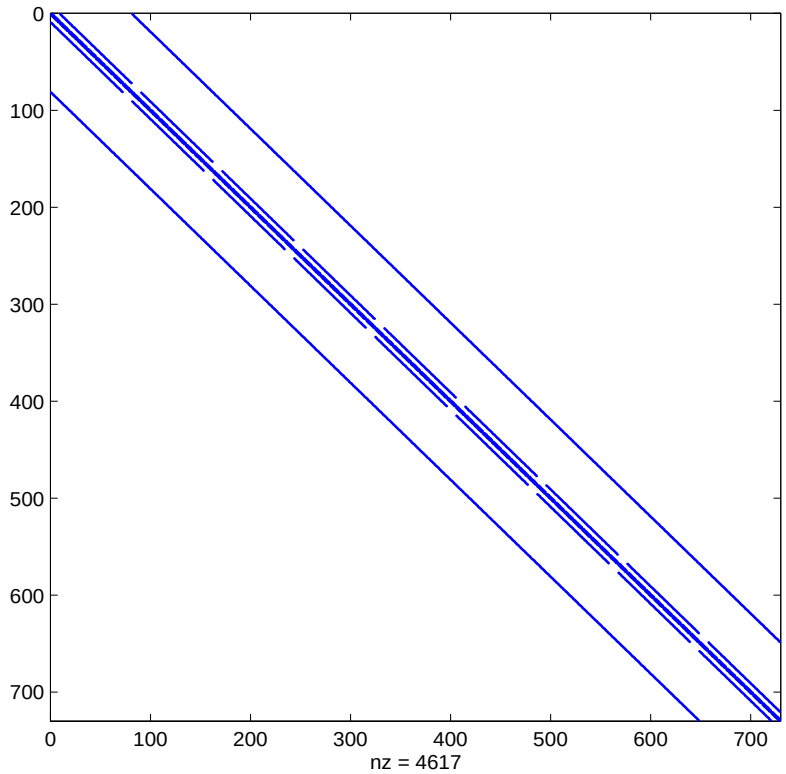


$$\lambda_{min} = 5.38\ 10^{-4}, \quad \lambda_{max} = 8.49\ 10^{-1},\ \kappa(A) = 1.58\ 10^{3}$$

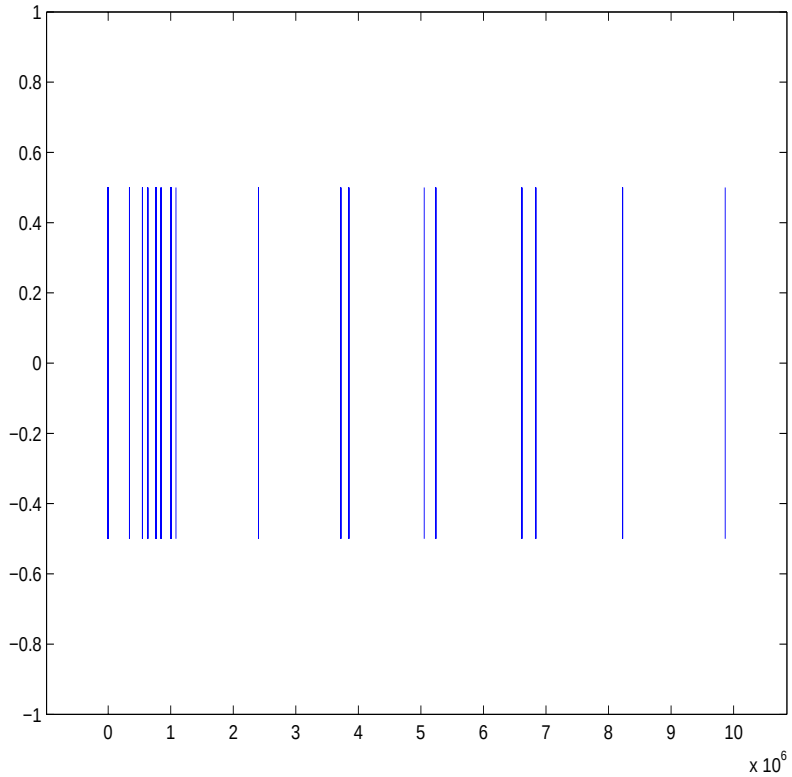


Eigenvalues of Nos4

- Nos7 $n = 729$


0
100
200
300
400
500
600
700
0
100
200
300
400
500
600
700
nz = 4617

$$\lambda_{min} = 4.15\ 10^{-3}, \quad \lambda_{max} = 9.86\ 10^{6},\ \kappa(A) = 2.37\ 10^{9}$$



Eigenvalues of Nos7

Results of CG

Poisson $m \times m$ mesh, $m = 30, \varepsilon = 10^{-6}$

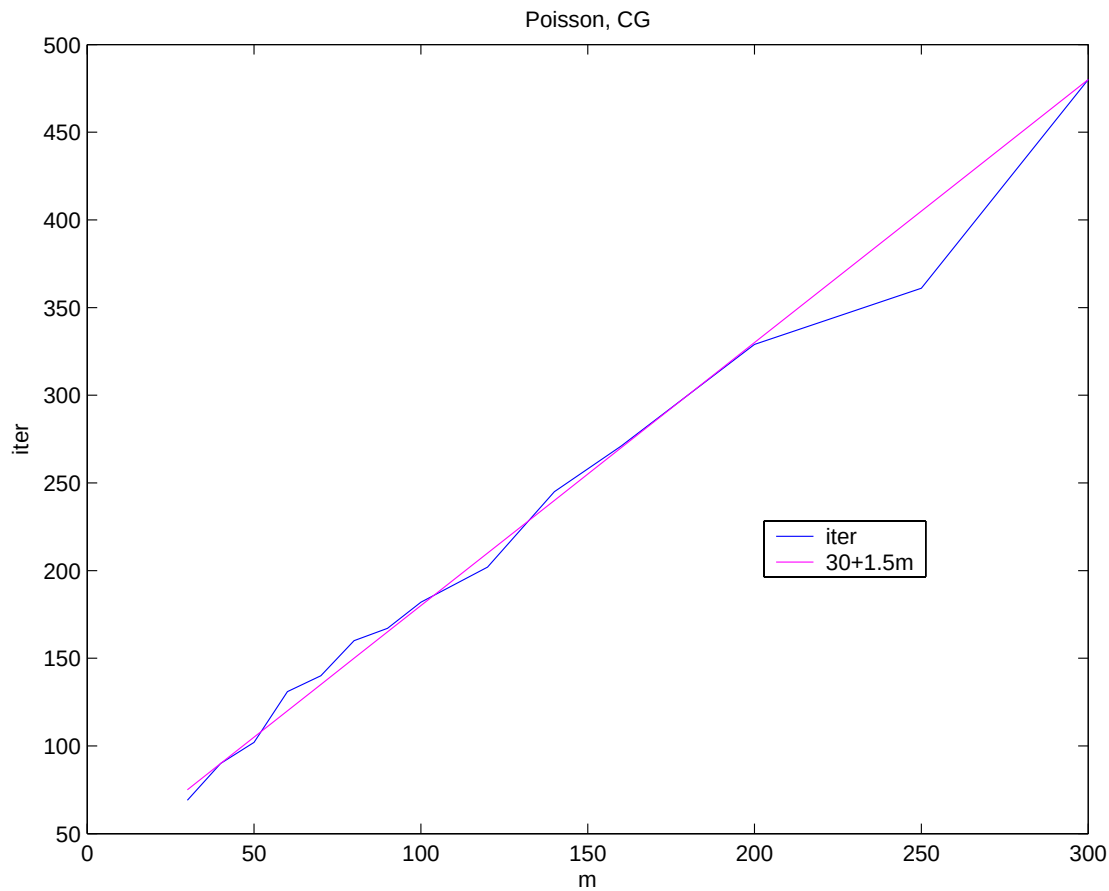
method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	79	$1.1628\ 10^{-5}$	$1.5768\ 10^{-5}$	0.062
diag	79	$1.1628\ 10^{-5}$	$1.5768\ 10^{-5}$	0.094
IC(0)	27	$1.2605\ 10^{-5}$	$2.2493\ 10^{-5}$	0.094
IC(l=2)	18	$9.8085\ 10^{-6}$	$3.1024\ 10^{-5}$	0.063
IC(l=3)	15	$6.6183\ 10^{-6}$	$1.3440\ 10^{-5}$	0.063
INV	13	$3.3922\ 10^{-6}$	$4.2056\ 10^{-6}$	-
SSOR $\omega = 1$	31	$6.6183\ 10^{-6}$	$1.3440\ 10^{-5}$	0.172
POL $deg = 1$	44	$1.2507\ 10^{-5}$	$1.8475\ 10^{-5}$	0.109
POL $deg = 2$	24	$8.7294\ 10^{-6}$	$1.2651\ 10^{-5}$	0.125
POL $deg = 3$	19	$1.6477\ 10^{-5}$	$2.9369\ 10^{-5}$	0.125
AINV $\tau = 0.25$	41	$4.5986\ 10^{-6}$	$6.8953\ 10^{-5}$	0.078
AINV $\tau = 0.1$	28	$3.9936\ 10^{-6}$	$6.1766\ 10^{-5}$	0.078
AINV $\tau = 0.05$	19	$8.6031\ 10^{-6}$	$1.5788\ 10^{-4}$	0.094
SAINV $\tau = 0.25$	41	$5.2743\ 10^{-6}$	$7.0618\ 10^{-5}$	0.078
ml 'gs,b,st,st' $\tau = 0.05$	4	$3.0250\ 10^{-6}$	$1.4427\ 10^{-5}$	0.093
ml 'ic,b,st,st' $\tau = 0.05$	4	$8.9591\ 10^{-8}$	$5.0452\ 10^{-7}$	0.063

Poisson $m \times m$ mesh, $m = 50, \varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	127	$2.6037\ 10^{-5}$	$8.4147\ 10^{-5}$	0.297
diag	127	$2.6037\ 10^{-5}$	$8.4147\ 10^{-5}$	0.375
IC(0)	39	$1.2263\ 10^{-5}$	$5.2022\ 10^{-4}$	0.375
IC(l=2)	26	$1.8554\ 10^{-5}$	$2.6769\ 10^{-4}$	0.25
IC(l=3)	22	$1.9601\ 10^{-5}$	$1.6566\ 10^{-4}$	0.235
INV	19	$1.6203\ 10^{-5}$	$2.5806\ 10^{-5}$	-
SSOR $\omega = 1$	46	$2.2925\ 10^{-5}$	$1.4882\ 10^{-4}$	0.687
POL $deg = 1$	70	$2.9105\ 10^{-5}$	$1.3250\ 10^{-4}$	0.438
POL $deg = 2$	49	$2.7637\ 10^{-5}$	$1.3848\ 10^{-4}$	0.484
POL $deg = 3$	38	$2.5000\ 10^{-5}$	$1.1778\ 10^{-4}$	0.515
AINV $\tau = 0.25$	65	$1.1334\ 10^{-5}$	$1.7105\ 10^{-4}$	0.328
AINV $\tau = 0.1$	41	$1.0740\ 10^{-5}$	$3.7991\ 10^{-4}$	0.359
AINV $\tau = 0.05$	30	$1.0079\ 10^{-5}$	$4.0654\ 10^{-4}$	0.422
SAINV $\tau = 0.25$	65	$1.2247\ 10^{-5}$	$2.0315\ 10^{-4}$	0.329
ml 'gs,b,st,st' $\tau = 0.05$	4	$6.0406\ 10^{-6}$	$3.2352\ 10^{-5}$	0.235
ml 'ic,b,st,st' $\tau = 0.05$	4	$2.3068\ 10^{-7}$	$1.5569\ 10^{-6}$	0.171

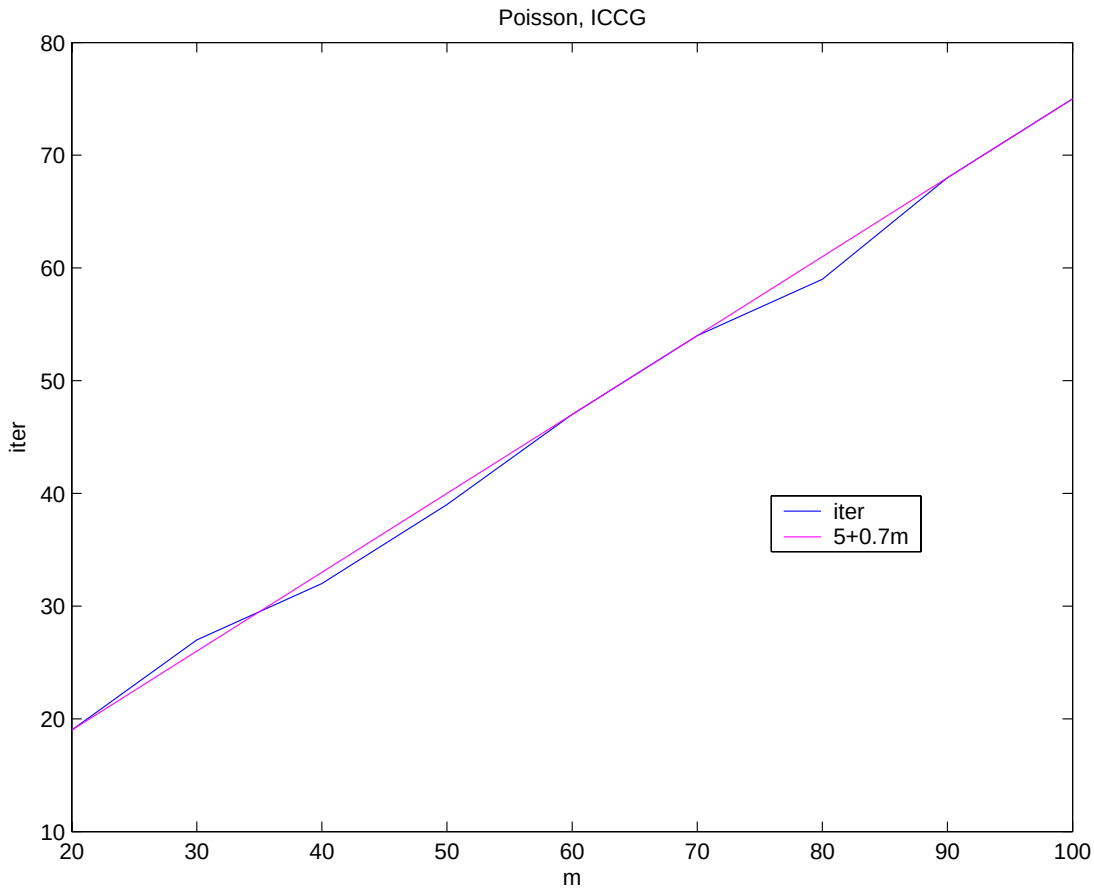
Number of iterations for CG as a function of m for an $m \times m$ mesh

$$\epsilon = 10^{-6}, b = 0, x^0 = rand$$



Number of iterations for ICCG as a function of m for an $m \times m$ mesh

$$\epsilon = 10^{-6}, b = 0, x^0 = rand$$



Pb12 $m \times m$ mesh, $m = 50, \varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	229	$2.8809\ 10^{-5}$	$7.0818\ 10^{-6}$	0.609
diag	229	$2.8809\ 10^{-5}$	$7.0818\ 10^{-6}$	0.656
IC(0)	20	$2.0240\ 10^{-6}$	$1.8357\ 10^{-4}$	0.187
IC(l=2)	8	$9.8018\ 10^{-6}$	$6.0588\ 10^{-6}$	0.079
IC(l=3)	8	$5.4194\ 10^{-6}$	$2.9146\ 10^{-6}$	0.078
INV	8	$5.3315\ 10^{-6}$	$2.8441\ 10^{-6}$	-
SSOR $\omega = 1$	79	$2.2503\ 10^{-5}$	$5.3162\ 10^{-6}$	1.172
POL $deg = 1$	128	$2.9154\ 10^{-5}$	$7.5585\ 10^{-6}$	0.796
POL $deg = 2$	91	$2.2249\ 10^{-5}$	$6.1957\ 10^{-6}$	0.907
POL $deg = 3$	70	$2.5175\ 10^{-5}$	$6.8154\ 10^{-6}$	0.953
AINV $\tau = 0.25$	31	$1.3073\ 10^{-6}$	$2.9591\ 10^{-5}$	0.297
AINV $\tau = 0.1$	25	$9.6974\ 10^{-7}$	$2.5045\ 10^{-5}$	0.328
AINV $\tau = 0.05$	24	$1.1940\ 10^{-6}$	$5.0621\ 10^{-5}$	0.375
SAINV $\tau = 0.25$	41	$1.3674\ 10^{-6}$	$3.5966\ 10^{-5}$	0.359
ml 'gs,b,st,st' $\tau = 0.05$	4	$5.9298\ 10^{-7}$	$5.0765\ 10^{-6}$	0.281
ml 'ic,b,st,st' $\tau = 0.05$	3	$3.7026\ 10^{-10}$	$8.8933\ 10^{-9}$	0.156

Pb14 $m \times m$ mesh, $m = 50, \varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	1551	$2.6211\ 10^{-5}$	$2.7909\ 10^{-5}$	3.859
diag	166	$2.4297\ 10^{-5}$	$2.3925\ 10^{-5}$	0.469
IC(0)	53	$1.0033\ 10^{-5}$	$1.4868\ 10^{-4}$	0.5
IC(l=2)	36	$1.5683\ 10^{-5}$	$5.1312\ 10^{-6}$	0.344
IC(l=3)	29	$1.8448\ 10^{-5}$	$7.1641\ 10^{-6}$	0.297
INV	26	$1.5417\ 10^{-5}$	$1.8905\ 10^{-6}$	-
SSOR $\omega = 1$	65	$2.0698\ 10^{-5}$	$3.0055\ 10^{-5}$	0.953
POL $deg = 1$	901	$2.1013\ 10^{-5}$	$2.7195\ 10^{-5}$	6.015
POL $deg = 2$	635	$2.7254\ 10^{-5}$	$3.3552\ 10^{-5}$	6.578
POL $deg = 3$	499	$2.9084\ 10^{-5}$	$4.3737\ 10^{-5}$	7.047
POL $deg = 1$ norm	92	$8.9127\ 10^{-6}$	$1.3304\ 10^{-4}$	0.593
POL $deg = 2$ norm	68	$1.0259\ 10^{-5}$	$1.4885\ 10^{-4}$	0.703
POL $deg = 3$ norm	65	$1.0591\ 10^{-5}$	$1.5000\ 10^{-4}$	0.969
AINV $\tau = 0.25$	87	$1.0230\ 10^{-5}$	$1.4061\ 10^{-4}$	0.469
AINV $\tau = 0.1$	54	$9.7391\ 10^{-6}$	$1.5163\ 10^{-4}$	0.469
AINV $\tau = 0.05$	37	$7.6959\ 10^{-6}$	$1.4915\ 10^{-4}$	0.532
SAINV $\tau = 0.25$	80	$9.6212\ 10^{-6}$	$2.3974\ 10^{-4}$	0.422
SAINV $\tau = 0.1$	51	$1.0737\ 10^{-5}$	$1.9716\ 10^{-4}$	0.453
SAINV $\tau = 0.05$	34	$7.7236\ 10^{-6}$	$8.6639\ 10^{-5}$	0.547
ml 'gs,b,st,st' $\tau = 0.05$	8	$3.1880\ 10^{-7}$	$3.3869\ 10^{-6}$	0.469
ml 'ic,b,st,st' $\tau = 0.05$	6	$7.5153\ 10^{-6}$	$3.8023\ 10^{-5}$	0.265

Pb26 $m \times m$ mesh, $m = 50, \varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	2013	$2.3420\ 10^{-5}$	$4.1623\ 10^{-5}$	4.688
diag	436	$2.5696\ 10^{-5}$	$3.1074\ 10^{-5}$	1.188
IC(0)	83	$2.0339\ 10^{-5}$	0.0013	0.766
IC(l=2)	28	$2.3474\ 10^{-5}$	$3.1773\ 10^{-5}$	0.266
IC(l=3)	24	$1.8063\ 10^{-5}$	$2.6646\ 10^{-5}$	0.25
INV	15	0.0032	0.0021	-
SSOR $\omega = 1$	214	$2.8148\ 10^{-5}$	$5.3301\ 10^{-5}$	3.078
POL $deg = 1$	1577	$2.9007\ 10^{-5}$	$4.1807\ 10^{-5}$	9.719
POL $deg = 2$	1262	$1.9681\ 10^{-5}$	$3.9558\ 10^{-5}$	12.141
POL $deg = 3$	1038	$2.6075\ 10^{-5}$	$4.4625\ 10^{-5}$	13.5
POL $deg = 1$ norm	2723	$1.9034\ 10^{-5}$	0.0015	17.532
AINV $\tau = 0.25$	103	$1.8260\ 10^{-5}$	0.0015	0.484
AINV $\tau = 0.1$	70	$1.7278\ 10^{-5}$	0.0011	0.5
AINV $\tau = 0.05$	47	$1.6103\ 10^{-5}$	$7.8652\ 10^{-4}$	0.531
SAINV $\tau = 0.25$	78	$1.8207\ 10^{-5}$	0.0021	0.359
SAINV $\tau = 0.1$	58	$1.7891\ 10^{-5}$	$9.6876\ 10^{-4}$	0.406
SAINV $\tau = 0.05$	40	$1.4228\ 10^{-5}$	$5.8608\ 10^{-4}$	0.438
ml 'gs,b,st,st' $\tau = 0.05$	23	$1.3891\ 10^{-5}$	0.0010	1.5
ml 'ic,b,st,st' $\tau = 0.05$	13	$0.609\ 10^{-5}$	$6.3135\ 10^{-4}$	0.172

Bcsstk01, $\varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	139	$3.8910\ 10^{-6}$	$7.4606\ 10^{-13}$	0.031
diag	48	$1.9818\ 10^{-6}$	$5.3852\ 10^{-15}$	0.015
IC(0)	16	$9.3495\ 10^{-7}$	$6.2030\ 10^{-14}$	0.015
SSOR $\omega = 1$	25	$2.2969\ 10^{-7}$	$9.7950\ 10^{-16}$	0.016
POL $deg = 1$	111	$9.3578\ 10^{-7}$	$1.6543\ 10^{-13}$	0.031
POL $deg = 2$	91	$3.8830\ 10^{-6}$	$2.9354\ 10^{-13}$	0.047
POL $deg = 3$	81	$1.1262\ 10^{-6}$	$2.0599\ 10^{-13}$	0.047
AINV $\tau = 0.25$	20	$2.1759\ 10^{-9}$	$7.9490\ 10^{-9}$	-
AINV $\tau = 0.1$	14	$2.6290\ 10^{-9}$	$1.4125\ 10^{-8}$	-
AINV $\tau = 0.05$	13	$1.3292\ 10^{-9}$	$8.7294\ 10^{-9}$	-
SAINV $\tau = 0.25$	16	$2.5931\ 10^{-9}$	$1.0420\ 10^{-8}$	-
SAINV $\tau = 0.1$	13	$2.4287\ 10^{-9}$	$1.0611\ 10^{-8}$	-
ml 'gs,b,st,st' $\tau = 0.05$	10	$1.0272\ 10^{-9}$	$1.0107\ 10^{-8}$	-
ml 'ic,b,st,st' $\tau = 0.05$	9	$1.9440\ 10^{-9}$	$5.6457\ 10^{-9}$	-

Bcsstk14, $\varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	14924	$2.1901\ 10^{-5}$	$1.5159\ 10^{-10}$	108.469
diag	457	$2.1686\ 10^{-5}$	$1.8488\ 10^{-11}$	3.156
IC(0) *	258	$1.5376\ 10^{-5}$	$8.5034\ 10^{-12}$	4.718
IC(lev=2) *	48	$1.7116\ 10^{-5}$	$9.1611\ 10^{-12}$	1.141
IC(lev=3)	33	$1.8114\ 10^{-5}$	$1.2675\ 10^{-11}$	0.766
SSOR $\omega = 1$	186	$2.4450\ 10^{-5}$	$2.3062\ 10^{-11}$	4.61
POL	too slow conv			
AINV $\tau = 0.25$	79	$3.0003\ 10^{-6}$	$1.7563\ 10^{-4}$	0.703
AINV $\tau = 0.1$	50	$3.4530\ 10^{-6}$	$4.8175\ 10^{-4}$	0.625
AINV $\tau = 0.05$	35	$2.5568\ 10^{-6}$	$4.2597\ 10^{-5}$	0.656
SAINV $\tau = 0.25$	77	$3.1263\ 10^{-6}$	$8.0122\ 10^{-5}$	0.703
SAINV $\tau = 0.1$	45	$3.4120\ 10^{-6}$	$7.6038\ 10^{-5}$	0.562
SAINV $\tau = 0.05$	32	$2.4114\ 10^{-6}$	$3.2010\ 10^{-5}$	0.578
ml 'gs,b,st,st' $\tau = 0.05$	69	$2.5627\ 10^{-6}$	$7.7924\ 10^{-4}$	8.375
ml 'ic,b,st,st' $\tau = 0.05$ *	non conv			

* Pb with the incomplete decomposition

Bcsstk27, $\varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	960	$1.7741\ 10^{-5}$	$9.2437\ 10^{-9}$	5.469
diag	223	$1.9009\ 10^{-5}$	$3.3812\ 10^{-9}$	1.312
IC(0)	17	$1.6208\ 10^{-7}$	$5.4478\ 10^{-6}$	0.266
IC(lev=2)	15	$1.7127\ 10^{-5}$	$1.7893\ 10^{-9}$	0.25
IC(lev=2)	9	$1.4454\ 10^{-5}$	$1.7395\ 10^{-9}$	0.156
POL	too slow conv			
AINV $\tau = 0.25$	104	$2.2961\ 10^{-7}$	$2.8573\ 10^{-6}$	0.719
AINV $\tau = 0.1$	95	$1.9541\ 10^{-7}$	$1.8153\ 10^{-6}$	1.032
AINV $\tau = 0.05$	44	$2.2123\ 10^{-7}$	$4.5417\ 10^{-6}$	0.719
SAINV $\tau = 0.25$	114	$1.9908\ 10^{-7}$	$1.5915\ 10^{-6}$	0.797
SAINV $\tau = 0.1$	60	$1.9753\ 10^{-7}$	$1.1930\ 10^{-6}$	0.531
SAINV $\tau = 0.05$	38	$2.2659\ 10^{-7}$	$2.8418\ 10^{-6}$	0.5
ml 'gs,b,st,st' $\tau = 0.05$	36	$1.7186\ 10^{-7}$	$5.5210\ 10^{-6}$	3.532
ml 'ic,b,st,st' $\tau = 0.05$	11	$2.0962\ 10^{-7}$	$4.2686\ 10^{-6}$	0.578

Nos4, $\varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	76	$5.9340\ 10^{-6}$	$5.9305\ 10^{-5}$	0.031
diag	70	$2.8677\ 10^{-6}$	$3.0437\ 10^{-5}$	0.016
IC(0)	20	$4.7516\ 10^{-6}$	$1.0189\ 10^{-5}$	-
IC(lev=2)	14	$3.8063\ 10^{-6}$	$5.9490\ 10^{-5}$	0.016
IC(lev=3)	12	$4.1614\ 10^{-6}$	$3.9330\ 10^{-5}$	-
SSOR $\omega = 1$	29	$3.8942\ 10^{-6}$	$4.0602\ 10^{-5}$	0.032
POL $deg = 1$	52	$2.8072\ 10^{-6}$	$2.5779\ 10^{-5}$	0.015
POL $deg = 2$	55	$3.3823\ 10^{-6}$	$3.4445\ 10^{-5}$	0.046
POL $deg = 3$	99	$3.8035\ 10^{-6}$	$2.1027\ 10^{-5}$	0.094
AINV $\tau = 0.25$	31	$4.4905\ 10^{-6}$	$8.5859\ 10^{-6}$	0.015
AINV $\tau = 0.1$	24	$3.9463\ 10^{-6}$	$8.5609\ 10^{-6}$	-
AINV $\tau = 0.05$	19	$2.3956\ 10^{-6}$	$8.5436\ 10^{-6}$	0.016
SAINV $\tau = 0.25$	32	$3.3688\ 10^{-6}$	$9.2407\ 10^{-6}$	0.016
SAINV $\tau = 0.1$	22	$8.9581\ 10^{-6}$	$2.6705\ 10^{-5}$	-
SAINV $\tau = 0.1$	17	$8.7502\ 10^{-6}$	$2.1135\ 10^{-5}$	0.016
ml 'gs,b,st,st' $\tau = 0.05$	16	$4.6129\ 10^{-6}$	$1.4803\ 10^{-5}$	0.062
ml 'ic,b,st,st' $\tau = 0.05$	10	$1.3330\ 10^{-6}$	$3.3351\ 10^{-6}$	0.031

Nos7, $\varepsilon = 10^{-6}$

method	nb it	$\|r^k\|$	$\|\epsilon^k\|$	time (s)
no prec	3776	$1.0649\ 10^{-5}$	$1.9434\ 10^{-4}$	4.266
diag	89	$9.3199\ 10^{-6}$	$6.3311\ 10^{-6}$	0.078
IC(0)	23	$2.2085\ 10^{-5}$	0.0018	0.062
IC(lev=2)	25	$3.2361\ 10^{-6}$	$5.8652\ 10^{-6}$	0.078
SSOR $\omega = 1$	38	$3.4097\ 10^{-6}$	$2.7747\ 10^{-6}$	0.156
POL $deg = 1$	2979	$1.3054\ 10^{-5}$	$1.9465\ 10^{-4}$	8.422
POL $deg = 2$	2543	$1.1969\ 10^{-5}$	$1.9432\ 10^{-4}$	10.156
POL $deg = 3$	2245	$9.9299\ 10^{-6}$	$9.1589\ 10^{-5}$	12.062
POL $deg = 1$ norm	1540	$3.6523\ 10^{-5}$	0.0017	3.922
POL $deg = 2$ norm	612	$3.6440\ 10^{-5}$	0.0017	2.375
POL $deg = 3$ norm	87	$2.1362\ 10^{-5}$	0.0010	0.422
AINV $\tau = 0.25$	60	$2.1879\ 10^{-5}$	0.0021	0.094
AINV $\tau = 0.1$	33	$3.6340\ 10^{-5}$	0.0013	0.078
AINV $\tau = 0.05$	27	$3.4963\ 10^{-5}$	0.0011	0.094
SAINV $\tau = 0.25$	62	$3.7568\ 10^{-5}$	0.0016	0.094
SAINV $\tau = 0.1$	33	$1.5978\ 10^{-5}$	0.0029	0.078
SAINV $\tau = 0.05$	26	$2.1718\ 10^{-5}$	0.0036	0.094
ml 'gs,b,st,st' $\tau = 0.05$	8	$1.6675\ 10^{-5}$	0.0042	0.203
ml 'ic,b,st,st' $\tau = 0.05$	7	$5.1063\ 10^{-7}$	$7.8618\ 10^{-4}$	0.125