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Domain Decomposition Methods for Time Dependent Problems

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Outline :

1) Problems to be solved

2) Sketch of the method

3) Decay of inverses

- 1D
- 2D

4) Description of the method

5) Numerical experiments

Problems to be solved

We are interested into solving linear systems arising from the implicit discretization of linear partial differential parabolic equations like,

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(a(x,y)\frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(b(x,y)\frac{\partial u}{\partial y}) = f$$

$$\text{in } \Omega \subset R^d, \quad d = 1 \quad \text{or} \quad d = 2$$

$$u \mid_{\partial\Omega} = 0, \quad u(x,0) = u_0(x)$$

The model problem will be

$$\frac{\partial u}{\partial t} - \Delta u = f$$

Ω is the unit square

For efficiency, we use an implicit scheme

Crank–Nicolson scheme

$$\frac{\partial u}{\partial t} + Lu = f$$

Finite difference discretization with $n+1$ points on each side of the square

$$h = \frac{1}{n+1}$$

$$L \Longrightarrow \frac{1}{h^2} A$$

A is a block tridiagonal matrix (of the steady state problem)

Time is discretized with a centered scheme : $t \in [0, T]$, step size $k = \Delta t$

$$2\frac{h^2}{k}(u^{m+1} - u^m) + Au^{m+1} + Au^m = h^2(f^{m+1} + f^m)$$

For every time step we have to solve the system

$$(2\frac{h^2}{k}I + A)u^{m+1} = 2\frac{h^2}{k}u^m - Au^m + h^2(f^{m+1} + f^m)$$

The advantages of the implicit scheme are :

– second order accuracy

– unconditional stability

Let $\theta = 2\frac{h^2}{k}$, the matrix of the system is

$$A_t = \theta I + A$$

A_t is a symmetric strictly diagonally dominant M–matrix

In 1D :

$$A_t = \begin{pmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{pmatrix},$$

$$a = 2 + \theta$$

In 2D :

$$A_t = \begin{pmatrix} D_t & -I & & & \\ -I & D_t & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & D_t & -I \\ & & & -I & D_t \end{pmatrix},$$

$$D_t = \begin{pmatrix} b & -1 & & & \\ -1 & b & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b & -1 \\ & & & -1 & b \end{pmatrix},$$

$$b = 4 + \theta$$

To solve the system, we use the **Conjugate Gradient method**

$$r^0 = b - Ax^0,$$

for $k = 0, 1, \ldots$ until convergence,

$$Mz^k = r^k,$$

$$\beta_k = \frac{(r^k, z^k)}{(r^{k-1}, z^{k-1})}, \ \beta_0 = 0,$$

$$p^k = z^k + \beta_k p^{k-1},$$

$$\alpha_k = \frac{(r^k, z^k)}{(Ap^k, p^k)},$$

$$x^{k+1} = x^k + \alpha_k p^k,$$

$$r^{k+1} = r^k - \alpha_k Ap^k.$$

We are looking for a parallel preconditioner M

Sketch of the method

We use a method which was proposed in a different setting by Y. Kuznetsov
New algorithms for approximate realization of implicit difference schemes
Sov. J. Numer. Anal. Math. Modelling v 3 n^0 2 (1988)

Main tools :

–1) Superposition of solutions
–2) Decay of elements of inverses

Superposition of solutions

Linearity

$$A_t x_1 = b_1, \quad A_t x_2 = b_2 \Longrightarrow A_t(x_1 + x_2) = b_1 + b_2$$

Figure

The problem is to know where to end the subdomains i.e. to analyse the decay of the solution when we put a δ function in a point of the mesh

We must have estimates on the decay of the entries of ${A_t}^{-1}$

Consider first the 1D case :

$$A_t = \begin{pmatrix} a & -1 & & & \\ -1 & a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & a & -1 \\ & & & -1 & a \end{pmatrix},$$

$$a = 2 + \theta$$

Decay of the elements of the inverse

It is easy to find the inverse of A_t.

There exists two sequences $\{u_i\}, \{v_i\}, i = 1, n$ such that

$$A_t^{-1} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 & \dots & u_1v_n \\ u_1v_2 & u_2v_2 & u_2v_3 & \dots & u_2v_n \\ u_1v_3 & u_2v_3 & u_3v_3 & \dots & u_3v_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1v_n & u_2v_n & u_3v_n & \dots & u_nv_n \end{pmatrix}.$$

We compute $\{u_i\}, \{v_i\}$ in the following way :
Consider a UL decomposition of A_t

$$A_t = UDU^T,$$

with

$$U = \begin{pmatrix} 1 & -\gamma_1 & & & \\ & 1 & -\gamma_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & -\gamma_{n-1} \\ & & & & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & \\ & & & d_{n-1} & \\ & & & & d_n \end{pmatrix}.$$

We have

$$d_n = a,$$

$$d_i = a - \frac{1}{d_{i+1}}, \quad \gamma_i = \frac{1}{d_{i+1}}, \quad i = n-1, \ldots, 1$$

For the Toeplitz case we can solve this recurrence with the following lemma :

Lemma

Let

$$\alpha_1 = a,$$

$$\alpha_i = a - \frac{1}{\alpha_{i-1}}, \quad i = 2, \ldots, n$$

then, if $a \neq 2$,

$$\alpha_i = \frac{r_+^{i+1} - r_-^{i+1}}{r_+^i - r_-^i},$$

where $r_\pm = \frac{a \pm \sqrt{a^2-4}}{2}$ are the two solutions of the quadratic equation $r^2 - ar + 1 = 0$.
If $a = 2$, then $\alpha_i = \frac{i+1}{i}$.

From this lemma, we get that for A_t,

$$d_{n-i+1} = \frac{r_+^{i+1} - r_-^{i+1}}{r_+^i - r_-^i}.$$

The $\{u_i\}, \{v_i\}$ are only defined up to a multiplicative constant; we take $u_1 = 1$.
Then, let

$$v = (v_1, \ldots, v_n)^T,$$

the first column of A_t^{-1} is v, so

$$A_t v = e_1,$$

where $e_1 = (1, 0, \ldots, 0)^T$.
With the help of the UL decomposition we solve this system.

$$DU^T v = e_1,$$

and hence,

Proposition

$$v_1 = \frac{1}{d_1},$$

$$v_i = \frac{1}{d_i d_{i-1} \cdots d_1}, \quad i = 2, \ldots, n$$

We have

$$v_i = \frac{r_+^{n-i+1} - r_-^{n-i+1}}{r_+^{n+1} - r_-^{n+1}}, \quad \forall i.$$

In particular

$$v_n = \frac{r_+ - r_-}{r_+^{n+1} - r_-^{n+1}}.$$

Let $u = (u_1, \ldots, u_n)^T$,
the last column of A_t^{-1} is $v_n u$ and therefore

$$v_n A_t u = e_n,$$

where $e_n = (0, \ldots, 0, 1)^T$.
To solve this system, it is easier to compute an LU decomposition of A_t:

$$A_t = L \Delta L^T,$$

$$L = \begin{pmatrix} 1 & & & & \\ -\varepsilon_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -\varepsilon_{n-2} & 1 & \\ & & & -\varepsilon_{n-1} & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \delta_1 & & & & \\ & \delta_2 & & & \\ & & \ddots & & \\ & & & \delta_{n-1} & \\ & & & & \delta_n \end{pmatrix}.$$

We have,

$$\delta_1 = a,$$

$$\delta_i = a - \frac{1}{\delta_{i-1}}, \quad \varepsilon_i = \frac{1}{\delta_i}, \quad i = 2, \ldots, n$$

In this particular case

$$\delta_i = d_{n-i+1}.$$

Clearly,

$$\Delta L^T u = \frac{1}{v_n} e_n.$$

Solving for u, we have

Proposition

$$u_n = \frac{1}{\delta_n v_n},$$

$$u_{n-i} = \frac{1}{\delta_{n-i} \cdots \delta_n v_n}, \quad i = 1, \ldots, n-1.$$

Then

Proposition

For the sequence u_i in A_t^{-1},

$$u_i = \frac{r_+^i - r_-^i}{r_+ - r_-}, \quad i = 1, \ldots, n.$$

The preceding results prove the following theorem :

Theorem

For $j \geq i$,

$$A_t^{-1}{}_{i,j} = u_i v_j = \frac{(r_+^i - r_-^i)(r_+^{n-j+1} - r_-^{n-j+1})}{(r_+ - r_-)(r_+^{n+1} - r_-^{n+1})},$$

where $r_\pm$ are the two solutions of the quadratic equation $r^2 - ar + 1 = 0$.
for $a = 2(\theta = 0)$, we have

$$A_t^{-1}{}_{i,j} = i \ \frac{n-j+1}{n+1}.$$

We want to characterize the decay of the elements of A_t^{-1} along a row, $\forall j$ we have

$$\frac{A_t^{-1}{}_{i,j}}{A_t^{-1}{}_{i,j+1}} = d_j,$$

A_t is diagonally dominant then the sequence d_i is such that

$$d_i > 1.$$

Theorem

The sequence ${A_t^{-1}}_{i,j}$ is a strictly decreasing function of j, for $j > i$.

$$\frac{u_i v_j}{u_i v_{j+1}} = \frac{r_+^{n-j+1} - r_-^{n-j+1}}{r_+^{n-j} - r_-^{n-j}},$$

$$\frac{u_i v_j}{u_i v_{j+1}} = \frac{r_+^{n-j+1}}{r_+^{n-j}} \left(\frac{1 - r^{n-j+1}}{1 - r^{n-j}} \right) > r_+ > 1$$

where $r = \frac{r_-}{r_+}$.
From this, we deduce the following result :

Theorem

$\forall\ i$,

$$u_i v_i > (r_+)^{j-i} u_i v_j.$$

That is to say

$$\frac{u_i v_j}{u_i v_i} \leq \epsilon \qquad \text{if} \qquad j - i \geq \frac{\log \epsilon^{-1}}{\log r_+}.$$

2D case

$$A_t = \begin{pmatrix} D_t & -I & & & \\ -I & D_t & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & D_t & -I \\ & & & -I & D_t \end{pmatrix},$$

$$D_t = \begin{pmatrix} b & -1 & & & \\ -1 & b & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b & -1 \\ & & & -1 & b \end{pmatrix},$$

$$b = 4 + \theta$$

For that special case (separable equations with constant coefficients), we can extend the 1D results :

There exists two sequences of matrices $\{U_i\}, \{V_i\}, i = 1, n$ such that

$$A_t^{-1} = \begin{pmatrix} U_1V_1 & U_1V_2 & U_1V_3 & \dots & U_1V_n \\ U_1V_2 & U_2V_2 & U_2V_3 & \dots & U_2V_n \\ U_1V_3 & U_2V_3 & U_3V_3 & \dots & U_3V_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_1V_n & U_2V_n & U_3V_n & \dots & U_nV_n \end{pmatrix}.$$

We can compute $\{U_i\}, \{V_i\}$ with block LU and UL decompositions.

However, for more general cases, it is better to do it the following way :

$$A_t = \begin{pmatrix} D_1 & -A_2^T & & & \\ -A_2 & D_2 & -A_3^T & & \\ & \ddots & \ddots & \ddots & \\ & & -A_{n-1} & D_{n-1} & -A_n^T \\ & & & -A_n & D_n \end{pmatrix}.$$

Still compute a block LU : $(\Delta+L)\Delta^{-1}(\Delta+L^T)$ where L is the block lower triangular part of A_t.

$$\begin{cases} \Delta_1 = D_1, \\ \Delta_i = D_i - A_i \ \Delta_{i-1}^{-1} \ A_i{}^T. \end{cases}$$

and a block UL : $(\Sigma + L^T)\Sigma^{-1}(\Sigma + L)$

$$\begin{cases} \Sigma_n = D_n, \\ \Sigma_i = D_i - A_{i+1}^T \ \Sigma_{i+1}^{-1} \ A_{i+1}. \end{cases}$$

Let's compute the j^{th} block column X of A_t^{-1}

To do this we do a twisted factorization of A_t :

$$(\Phi + \mathcal{L})\Phi^{-1}(\Phi + \mathcal{L}^T)$$

We have

$$\Phi_i = \Delta_i, \quad i = 1, \ldots, j-1$$

$$\Phi_i = \Sigma_i, \quad i = n, \ldots, j+1$$

$$\Phi_j = D_j - A_j \ \Delta_{j-1}^{-1} \ A_j{}^T - A_{j+1}^T \ \Sigma_{j+1}^{-1} \ A_{j+1}$$

Figure

Then, we can solve the system for X :

The diagonal term is

$$X_j = \Phi_j^{-1}$$

Above the diagonal, we have

$$X_{j-l} = \Delta_{j-l}^{-1}\ {A_{j-l+1}}^T \cdots \Delta_{j-1}^{-1}\ {A_j}^T \Phi_j^{-1}$$

Below the diagonal, we have

$$X_{j+l} = \Sigma_{j+l}^{-1}\ A_{j+l} \cdots \Sigma_{j+1}^{-1}\ A_{j+1} \Phi_j^{-1}$$

Now, go back to the particular case of the model problem

$$\Delta_i \longrightarrow \Delta$$

$$\Sigma_i \longrightarrow \Sigma$$

$$\Delta = \Sigma$$

Asymptotically, the diagonal term of A_t^{-1} is

$$(2\ \Delta - D)^{-1}$$

In summary, to have an estimate of the decay :

Let

$$D_t = \begin{pmatrix} b & -1 & & & \\ -1 & b & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b & -1 \\ & & & -1 & b \end{pmatrix},$$

$$b = 4 + \theta$$

Compute the limit Δ of

$$\Delta_1 = D_t$$

$$\Delta_i = D_t - \Delta_{i-1}^{-1}$$

$$\Delta + \Delta^{-1} = D_t$$

Look at

$$(2\ \Delta - D_t)^{-1}$$